Forcing with Minimal Counterexamples

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Setup

We introduce the technique of *forcing with minimal counterexamples* and present some niceness properties of such iterations. The basic idea is simple: We force with all small, minimal counterexamples to some forcing axiom, and do the same in each stage of an iteration. We hope that no counterexamples are left if we do this often enough; naturally, "often enough" means of large cardinal length.

Definition 1. *Let $\{\mathbb{P}_\alpha, \alpha < \lambda\}$ be a set of forcing notions. The* **lottery sum** *of the $\mathbb{P}_\alpha$ is their disjoint union $\mathbb{P}$ with a new $\mathbb{1}$ such that $\mathbb{1} > p$ for all $p \in P_\alpha$, $\alpha < \lambda$.*

Definition 2. *Let $\mathcal{A}$ be a forcing axiom, i.e., a statement of the form "for all forcing notions $\mathbb{P}$, $\varphi(\mathbb{P})$" for some statement φ. Let κ be some ordinal. The* **counterexamples to $\mathcal{A}$ iteration with length** *κ is the countable support iteration of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa)$, where $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha$ are defined by induction: Let $\dot{\mathbb{Q}}_\alpha$ be a hereditarily minimal $\mathbb{P}_\alpha$-name for the lottery sum of all counterexamples to $\mathcal{A}$ of minimal hereditary size smaller κ.*

Remark 3. *This definition could be varied by defining each stage as* all *(semi)proper forcings in H_λ where $\lambda < \kappa$ is the hereditary size of a hereditarily minimal counterexample to $\mathcal{A}$.*

Counterexamples iterations of large cardinal length are relatively well-behaved. In particular this includes a number of "smallness" conditions, e.g., hereditary size and chain conditions.

Theorem 4. *Let $\mathbb{P}_\kappa$ be a counterexamples iteration. If κ is inaccessible and $\alpha < \kappa$, then $|\mathbb{P}_\alpha| < \kappa$.*

Corollary 5. *Let $\mathbb{P}_\kappa$ be a counterexamples iteration. If κ is inaccessible and $\alpha < \kappa$, then $\mathbb{P}_\alpha \in H_\kappa$.*

Corollary 6. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a counterexamples iteration, then it satisfies the κ-cc.*

Corollary 7. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a counterexamples iteration, it has at most cardinality κ.*

Corollary 8. *We can modify a counterexamples iteration $\mathbb{P}_\kappa$ to add Cohen reals cofinally often and collapse each cardinal $\lambda < \kappa$ to ω_1. If κ is inaccessible and $\mathbb{P}_\kappa$ preserves ω_1, then it forces $2^\omega = \kappa = \aleph_2 = 2^{\omega_1}$.*

Remark 9. *All of the above generalizes to* revised countable supports *instead of countable supports.*

We will primarily deal with *proper* forcings. Recall the following notions / facts:

Definition 10. *A condition p of some forcing $\mathbb{P}$ is called $(M, \mathbb{P})$*-**generic** *iff for every maximal antichain $A \in M$, $A \cap M$ is predense below p.*

Lemma 11. *$\mathbb{P}$ is proper iff for every regular λ such that $\mathbb{P} \in H_\lambda$ there is a club $C \subseteq [H_\lambda]^\omega$ of countable elementary submodels $M \prec (H_\lambda, \in, <, \mathbb{P}, \dots)$ where $<$ is some fixed well-ordering of H_λ such that*

$$\forall M \in C\ \forall p \in M \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-generic.}$$

Lemma 12. *Let $\mathbb{P}$ be a forcing notion, μ be a regular cardinal with $\mathbb{P} \in H_\mu$. Then $\mathbb{P}$ is proper iff there is a club $C \subseteq [H_\mu]^\omega$ of countable elementary submodels of H_μ such that*

$$\forall M \in C\ \forall p \in M\ \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-generic.}$$

Theorem 13. *Countable support iterations of proper forcings are themselves proper.*

Lemma 14. *If $\mathbb{P}$ is an Axiom A forcing notion, then $\mathbb{P}$ is proper.*

We also make frequent use of the following fact.

Lemma 15. *If κ is regular and $\mathbb{P} \subseteq H_\kappa$ satisfies the κ-cc: If $p \Vdash \dot{x} \in H_\kappa$, there is $\ddot{x} \in H_\kappa$ with $p \Vdash \dot{x} = \ddot{x}$.*

PFA from a supercompact

We restate Baumgartner's classical proof of the consistency of PFA given a supercompact cardinal within the framework of forcing with minimal counterexamples. Most notably, we do not require a Laver function or any other kind of fast-growing function.

Axiom 16 (Proper Forcing Axiom (PFA))**.** *If $(\mathbb{P}, <)$ is a proper forcing notion and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$, is a collection of dense subsets of $\mathbb{P}$, then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.*

Definition 17. *A cardinal κ is called λ*-**supercompact** *for some cardinal $\lambda \geq \kappa$ iff there is a model M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$ and $M^\lambda \subseteq M$.*
A cardinal κ is called **supercompact** *if it is λ-supercompact for all $\lambda \geq \kappa$.*

Lemma 18. *Let M be a transitive model with $\mathrm{Ord} \subseteq M$, $\mathbb{P} \in M$ a λ^+-cc forcing notion, G some $\mathbb{P}$-generic filter on M and λ a cardinal. In $V[G]$, if $V \models M^\lambda \subseteq M$ then $M[G]^\lambda \subseteq M[G]$.*

Lemma 19. *Let λ be a cardinal and $M^\lambda \subseteq M$ for some model M with $\mathrm{Ord} \subseteq M$. Then $H^M_{\lambda^+} \supseteq H_{\lambda^+}$.*

Theorem 20. *If κ is λ-supercompact, then the counterexamples to* PFA *iteration $\mathbb{P}_\kappa$ forces that* PFA *holds for all proper forcings $\mathbb{P}$ with $2^{|\mathbb{P}|} \leq \lambda$.*

Proof. We closely follow Baumgartner's proof. Let $j \colon V \to M$ be a λ-supercompactness embedding, i.e., $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $M^\lambda \subseteq M$. Assume the theorem is false. Let $p \in \mathbb{P}_\kappa$ force the failure of PFA. Suppose that G, $p \in G$, is $\mathbb{P}_\kappa$-generic over V. We work in $V[G]$. Let $\mathbb{P}$ be a proper forcing violating PFA with $2^{|\mathbb{P}|} \leq \lambda$ of minimal hereditary size. Let $\mathcal{D}$ be a collection of $\aleph_1$ many dense sets in $\mathbb{P}$ witnessing this. We know that $\mathbb{P} \in M[G]$ by Lemma 19, since $M[G]^\lambda \subseteq M[G]$ by Lemma 18.

Observation. *In $M[G]$, $\mathbb{P}$ violates* PFA, *is of minimal hereditary size with that property and $\mathbb{P} \in H_{j(\kappa)}$.*

Argument. We know $|\mathrm{TC}(\mathbb{P})| = |\mathbb{P}|$ by the minimality of $\mathbb{P}$. We now show that $\mathbb{P}$ is proper in $M[G]$. Let $\mu = (|\mathbb{P}|)^+$. Since $|\mathrm{TC}(\mathbb{P})| = |\mathbb{P}| < \mu$, $\mathbb{P} \in H_\mu$. Choose a club $C \subseteq [H_\mu]^\omega$ witnessing that $\mathbb{P}$ is proper in $V[G]$. Note that $|C| \leq |H_\mu| \leq 2^{<\mu} \leq 2^{|\mathbb{P}|} \leq \lambda$. Therefore 19, $C \in M[G]$ and hence $\mathbb{P}$ is proper in $M[G]$.
Furthermore we know that the set $\mathcal{D} = (D_\alpha \mid \alpha < \omega_1)$ in $V[G]$ in $M[G]$ by $M[G]$'s closure properties. $|\mathcal{D}|^{M[G]} = \aleph_1^{M[G]}$, since P_κ is proper (as a countable support iteration of proper forcing notions), i.e., $M[G]$ and $V[G]$ have the same $\aleph_1$.
Also, $|\mathrm{TC}(\mathbb{P})| < \lambda < j(\kappa)$, so $\mathbb{P} \in H_{j(\kappa)}$. Finally, if there were a hereditary smaller forcing notion in $M[G]$, it would be in $V[G]$ and contradict the hereditary minimality of $\mathbb{P}$. □

In M, the forcing $j(\mathbb{P}_\kappa)$ is a countable support iteration of length $j(\kappa) > \lambda$ and $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, since $\mathrm{crit}(j) = \kappa$ (i.e., $j \restriction H_\kappa = \mathrm{id}$ while $\mathbb{P}_\alpha \in H_\kappa$ for all $\alpha < \kappa$). There is a condition $q \leq p^\frown \mathbb{1}_{j(\kappa)}$ choosing $\mathbb{P}$ from the lottery sum in the κ-th step. Then, below q, $\mathbb{P}_{j(\kappa)}$ factors into $(\mathbb{P}_\kappa * \mathbb{P}) * \dot{\mathbb{P}}_{\kappa, j(\kappa)}$. Let H be $\mathbb{P}$-generic over $V[G]$ and I be $\dot{\mathbb{P}}^{G*H}_{\kappa, j(\kappa)}$-generic over $V[G * H]$. We now work in $V[(G * H) * I]$. Consider:

$$j^* \colon V[G] \to M[(G * H) * I], j^*(\sigma^G) = j(\sigma)^{(G*H)*I}.$$

It is easy to see that j^* is well-defined, elementary and extends j.
Notice that $j^* \restriction \mathbb{P} \in M[G]$, since $|\mathbb{P}| < \lambda$. $H \subseteq \mathbb{P}$ and therefore $j^*[H] \in M[(G * H) * I]$. Also, $V[G]$ and $M[(G * H) * I]$ agree on $\aleph_1$, i.e., $j^*(\omega_1) = \omega_1$, so $j^*(\mathcal{D}) = \{j^*(D) \mid D \in \mathcal{D}\}$. H is $\mathbb{P}$-generic in $V[G]$, in particular it intersects every $D \in \mathcal{D}$. Thus for every $D \in \mathcal{D}$ there is some $x_D \in H$ such that $V[G] \models x_D \in D$, so by elementarity, $M[(G * H) * I] \models j^*(x_D) \in j^*(D)$.
Therefore the filter on $j^*(\mathbb{P})$ generated by $j^*[H]$ in $M[(G * H) * I]$ intersects every $D \in j^*(\mathcal{D})$, i.e., it is $(j^*(\mathbb{P}), j^*(\mathcal{D}))$-generic. Hence, by elementarity, there is a $(\mathbb{P}, \mathcal{D})$-generic filter in $V[G]$. □

Corollary 21. (Baumgartner) PFA *is consistent relative to the existence of a supercompact cardinal.*

AAFA($\mathfrak{c}$) from a weakly compact

We now show that forcing with minimal counterexamples applies to smaller large cardinals. We reproduce a result by James Baumgartner, cf. [Bau84, Theorem 9.2].

Axiom 22 (Axiom A Forcing Axiom)**.** AAFA *is the restriction of* PFA *to Axiom A forcings.* AAFA($\mathfrak{c}$) *is the restriction of* AAFA *to forcings of size $\leq 2^\omega$.*

Lemma 23. *A cardinal κ with $\kappa^{<\kappa} = \kappa$ is* **weakly compact** *iff it satisfies the* Hauser property*: for each κ-model M there is a transitive N and an elementary embedding $j : M \to N$, $\mathrm{crit}(j) = \kappa$ with $j, M \in N$. (This is due to Kai Hauser [Hau91, Theorem 1.3])*

Theorem 24. (Baumgartner) *If κ is weakly compact, then an enhanced counterexamples to* AAFA($\mathfrak{c}$) *iteration $\mathbb{P}_\kappa$ forces* AAFA($\mathfrak{c}$) *with $\mathfrak{c} = \aleph_2$.*

Proof. Suppose the theorem is false. We can make $\mathbb{P}_\kappa$ force $\kappa = 2^\omega = \aleph_2$. Let $p \in \mathbb{P}_\kappa$ such that p forces that $(\dot{\mathbb{Q}}, \dot{\mathcal{D}})$ is a hereditarily minimal counterexample to AAFA($\mathfrak{c}$) Let $\dot{\mathcal{A}}$ be a name for a sequence of partial orders on $\dot{\mathbb{Q}}$ witnessing Axiom A. Let G be a generic filter with $p \in G$.
We may w.l.o.g. assume that $p \Vdash \dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq \kappa$. Thus by Lemma 15 we can suppose that $\dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq H_\kappa$. Let λ be large enough such that H_λ knows that $\dot{\mathbb{Q}}$ is a name for an Axiom A forcing as witnessed by $\dot{\mathcal{A}}$. Let $X \prec H_\lambda$ with $H_\kappa \subseteq X$, $\mathbb{P}_\kappa, \dot{\mathbb{Q}}, \dot{\mathcal{D}}, \dot{\mathcal{A}}, \kappa \in X$, $X^{<\kappa} \subseteq X$ and $|X| = \kappa$.
Let $X \to M$ be the Mostowski collapse of X, then M is a κ-model. Notice that since $\dot{\mathbb{Q}} \subseteq H_\kappa \subseteq X$ is in the transitive part of X, $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}} \in M$. Likewise $\dot{\mathcal{D}} \in M$, $\dot{\mathcal{A}} \in M$ and $\mathbb{P}_\kappa \in M$. Now let $j : M \to N$ be a κ-weakly compactness embedding.
Now work in $V[G]$. Note that $M[G] \in N[G]$ by the Hauser property and since $N[G]$ is transitive it contains all the sets we require (since we put them in $M[G]$). It is easy to check that $\mathcal{A} = (\leq_n)_{n \in \omega} \in M[G] \in N[G]$ also witnesses that $\mathbb{Q}$ satisfies Axiom A in $N[G]$. By the usual arguments, $N[G]$ then also knows that $\mathbb{Q}$ is a hereditary minimal counterexample to AAFA($\mathfrak{c}$). Also $|\mathrm{TC}(\mathbb{Q})| < \kappa^+ \leq j(\kappa)$.
Now consider $j(\mathbb{P}_\kappa)$. $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$ which in turn is an iteration of length $j(\kappa)$. Hence we can find some $q \leq p^\frown \mathbb{1}_{j(\kappa)}$ that chooses $\mathbb{Q}$ from the lottery sum in the κ-th step, i.e., below q, $j(\mathbb{P}_\kappa)$ factors into $\mathbb{P}_\kappa * \mathbb{Q} * \mathbb{R}$. Let H be $\mathbb{Q}$-generic over $V[G]$ and I be $\mathbb{R}$-generic over $V[G * H]$. Now work in $V[G * H * I]$.
As in Theorem 20, j lifts to an embedding $j^* : M[G] \to N^* = N[G * H * I]$ by mapping $j^*(\sigma^G) = j(\sigma)^{G*H*I}$.
Because $j^*, H \in N^*$, the set $j^*[H]$ is an element of N^* and generates a filter on $j^*(\mathbb{Q})$. Because H is $\mathbb{P}$-generic over M, for each $D \in \mathcal{D}$, there is some $x_D \in D \cap H$. Hence, by elementarity, $N^* \models j^*(x_D) \in j^*(D)$. Thus the filter generated by $j^*[H]$ is $(j^*(\mathbb{Q}), j^*(\mathcal{D}))$-generic. Again by elementarity, there must be a $(\mathbb{Q}, \mathcal{D})$-generic filter in $M[G]$. This filter would also be in $V[G]$ and contradict that $\mathbb{Q}$ is a counterexample to AAFA($\mathfrak{c}$). □

Remarks and References

So far we were also able to apply forcing with minimal counterexamples to the following results: BPFA from a reflecting cardinal[GS95], $\mathrm{MA}_{\aleph_1}$ and PFA^- from an inaccessible cardinal, $\mathrm{PFA}_\mathfrak{c}$ from a strongly unfoldable cardinal[HJ09, Theorem 6]/[Miy98, Theorem 3.1], PFA($\mathfrak{c}$) from a Σ^2_1-indescribable cardinal[NS08], and RA(proper) from an uplifting cardinal[Joh10]. All results generalize to their semi-proper variants using a RCS iteration of minimal counterexamples. We are confident that the method would also reproduce the main results of [NS08], [HJ09] and [Miy98]. The proofs we have found so far give rise to new large cardinal notions and new hierarchies of fragments of PFA. We are currently exploring these possibilities.
A very similar approach is the *proper lottery preparation* due to Hamkins and Johnstone[HJ09], independently introduced as *universal iteration* by Neeman and Schimmerling[NS08]. However, their approach also always requires some kind of fast-growing/set-anticipating function.

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