Forcing Axioms through Iterations of Minimal Counterexamples

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1 Introduction

1.1 Topic and scope

The topic of this thesis was proposed by Philipp Schlicht and based on work by Arthur Apter [Apt05] on removing Laver functions from the consistency proof of PFA. We conjectured that the forcing iteration Apter found would – with some subtle changes – generalize to different large cardinals and forcing axioms.

The primary goal of this thesis was to show that such modified iterations work at all. To accomplish that goal, we resolved to show a plurality of possible applications. Since the technique was first imagined with PFA in mind, most applications we found are centered around PFA, related axioms and fragments thereof. Nevertheless, we included Johnstone's *Resurrection Axiom* in Section 4.10 to show that the method generalizes to axioms that are farther removed from PFA.

Furthermore, to show that the method is not lacking in flexibility and generality compared with the classical arguments, we develop a theory of *revised countable supports*. We then quickly illustrate how our novel proofs of PFA and BPFA generalize to their semiproper analogues, just as the classical arguments do.

Developing our overarching framework, we tried to achieve a technique as general as possible. Now, indeed, we have a modular scheme for defining appropriate iterations for many forcing axioms while electively being able to both firmly control the size of the continuum or to leave it some leeway. However, we were not able to make any use of that treatment; most applications will force the continuum to be $\aleph_2$ whether or not we explicitly include this in our iteration.

We will present a total of ten applications. Furthermore, since the iterations in the applications are very similar, the proofs share a noticeable degree of similarity as well. Hence the actual proofs are quite short, since we could generalize many techniques to Lemmas now found in the preliminary sections. We expand on these considerations in Section 5.1.

We would hope that these advantages would enable us to derive some yet undiscovered consistency proofs, but, despite some efforts to improve on known results, we stayed within the scope of already known theorems. Nevertheless, we believe that many proofs found in this thesis are worth considering in their own right. The most notable difference from the classical arguments is that bookkeeping devices (usually fast-growing functions), applied in previous research on these axioms, are not required.

Of course there still is a great variety of axioms and their fragments that we did not apply our method to. However, we strongly believe that the method also applies to axioms such as PFA^+ and the like; in particular if their known proofs involve some variation of Baumgartner's argument for PFA. For example, we are confident that we could reproduce the proofs of the main results in [HJ09] and [NS08]. Aside from the general applicability, many open questions remain, see Section 5.3.

Besides discussing our novel techniques, we desired to provide a work that is, as far as reasonable and on a certain level, mathematically complete in itself. This sometimes lead to lemmas and proofs that may seem overly verbose. On the other hand, whenever an argument proved to be too far removed from the principal topic of this thesis to cite in completeness, we made sure to give a reference to either a commonplace literary work or a scholarly paper we checked to be commonly available. The section on revised countable supports is unfortunately particularly lacking in this regard, but we restricted ourself to cite only one quite verbose source, and prove everything not present there anew. Furthermore, we made sure to always include a reference whenever we basically just restated a proof found elsewhere.

Further considerations, including some musings on the meaning of this work within the broader scope of Set Theory and Mathematics itself, can be found in Section 5. Some analysis on the nature and structure of the proofs and results developed in this thesis can also be found there, as we thought it best to give these discussions after the reader had a chance to review the actual contents of this thesis.

1.2 Related work

The paper that inspired our approach is "Removing Laver Functions from Supercompactness Arguments" by Arthur Apter [Apt05] who considered iterated lottery sums of counterexamples of *minimal rank*. By Philipp Schlicht's insightful suggestion, we instead consider lottery sums of counterexamples of minimal *hereditary size*. Nevertheless, this thesis can be seen as the logical continuation of Apter's groundwork.

Apter, in turn, based his argument on the *lottery preparation* conceived by Hamkins [Ham98]. This approach was further extended by Hamkins and Johnstone [HJ09] to the *proper lottery preparation*, earlier called *PFA lottery preparation*. Much additional work with this iteration has been done, see, e.g., [Joh09], [Joh10].

Definition 1.1. [HJ09, Definition 1]. *Let κ be a cardinal and $f : \kappa \to \kappa$ be a partial function. The* **proper lottery preparation** *of κ relative f is the countable support iteration of length κ where at stages $\alpha \in \operatorname{dom} f$ we force with the lottery sum of all proper forcings in $H^{G_\alpha}_{f(\alpha)}$.*

The lottery preparations differ from our iterations in that they require some sort of fast-growing function to work properly. In contrast, one of our main ambitions was to remove any need for such functions from our method. Nevertheless, our approach seems to work well wherever the proper lottery preparation can be used. We illustrate this observation in our applications where we reproduce some of Johnstone's results.

The proper lottery preparation was independently described as the *universal iteration* by Neeman and Schimmerling [NS08, Definition 24]. They remark that their work is also inspired by Hamkins' original lottery preparation. Neeman and Schimmerling also require some fast-growing functions.

In our exposition we reproduce certain special cases of the main theorems by Neeman and Schimmerling resp. Hamkins and Johnstone. We believe that our proofs strongly indicate that fast-growing functions could be removed from their arguments altogether.

Going in a different direction, in Section 5.2 we note that our general method gives rise to hierarchies of fragments of PFA. Notable work in this direction has been done by Miyamoto [Miy98]. The proper lottery preparation would in principle also work for the hierarchies we describe, but one would require another general result stating that the large cardinal hierarchies we use carry fast functions.

Yet another area of related work is the characterization of large cardinals by certain embedding properties. While we did not contribute anything to this topic, we made frequent use of some related results. Notable is Hauser's [Hau91] work on indescribable cardinals. Furthermore, Hamkins [Ham02] has compiled an impressive and comprehensive collection of such characterizations and related Laver-like diamond principles.

1.3 Notation

We made an effort to only use standard notation, as far as there is an agreement on what is standard. Whenever we used a convenient shorthand or a notion people might disagree on, it should be listed here. We assume that the reader has a reasonably firm background in *Forcing*, in particular in iterations of forcing notions.

We will, for ease of notation, always force over V as the ground model. All philosophical issues this might raise are avoided by assuming that the model we call V is a countable ground model satisfying enough ZFC in some larger, unnamed, set universe. All definitions and proofs are then carried out within the model we call V, whatever its shape.

Notation 1.2 ($\subseteq_{<\omega}$)**.** *We say $A \subseteq_{<\omega} B$ iff A is a* finite subset *of B.*

Notation 1.3 (Continuum)**.** $\mathfrak{c} = 2^\omega$ *is the cardinality of the continuum.*

Notation 1.4 ($^\frown$)**.** *Let $\alpha < \beta$. If $p : \alpha \to V$ and $q : \beta \to V$ are functions, then the* continuation *of p along q, $p^\frown q : \beta \to V$, is defined by $(p^\frown q)(i) = p(i)$ if $i < \alpha$ and $(p^\frown q)(i) = q(i)$ otherwise.*

If $p : \alpha \to V$ and $q : \beta \setminus \alpha \to V$ are functions, then the concatenation *of p and q, $p^\frown q : \beta \to V$, is defined by $(p^\frown q)(i) = p(i)$ if $i < \alpha$ and $(p^\frown q) = q(i)$ otherwise.*

We abuse notation on $^\frown$. The two uses are always clear from the context.

Notation 1.5 (Forcing Notions)**.** *A* Forcing Notion*, respectively* Notion of Forcing*, or just* Forcing *is a preordered set $\mathbb{P}$, i.e., we do not require antisymmetry. We order forcing notions descendingly, i.e., $\mathbb{1}$ is the largest element and "stronger" conditions are smaller w.r.t. the order.*

If we write a forcing notion $\mathbb{P}_\kappa$ we mean that it is a forcing iteration of length κ.

Notation 1.6 (Names)**.** *Dotted variables, e.g., $\dot{a}$, $\ddot{a}$, are always* names*, checked variables, e.g., $\check{a}$, are always* canonical names*.*

Notation 1.7 (supp)**.** *If p is a condition in an iterated forcing,* supp(p) *denotes the* support *of p;* $\operatorname{supp}(p) = \{\beta \in \operatorname{dom}(p) \mid p \restriction \beta \nVdash p(\beta) = \mathbb{1}\}$.

Notation 1.8 (Compatible)**.** *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$ and $A \subseteq \mathbb{P}$. We say $p \leq A$ iff for all $a \in A$, $p \leq a$, and say that A is* compatible *iff there is some $q \in \mathbb{P}$ with $q \leq A$.*

Notation 1.9 (Genericity). *Let M be a ground model and $\mathbb{P} \in M$ be a forcing notion. We call a filter $G \subseteq \mathbb{P}$ that meets every dense $D \in M$ either* M-generic on $\mathbb{P}$ *or* $\mathbb{P}$-generic over M. *We also might leave out specifying M or $\mathbb{P}$ if this information is clear from the context.*

2 Preliminaries

2.1 Hereditary sets

In this section, we briefly review basic aspects of the theory of hereditary sets. In particular how they interact with the forcing relation; we show that the hereditary size of names may thought to be the same as that of their evaluations in a generic extension. Building on that, we arrive at the central result (Theorem 2.10), stating that under reasonable conditions the forcing relation commutes with "$H_\kappa \models$", i.e., that H_κ knows everything that will be forced about it.

Definition 2.1. *A set A has* **hereditary size** *α iff $|\mathrm{TC}(A)| = \alpha$.*

Definition 2.2. *H_λ is the set of all sets of hereditary size smaller λ, viz., $H_\lambda = \{a \mid |\mathrm{TC}(a)| < \lambda\}$.*

Remark 2.3. *This definition makes sense if λ is not a cardinal, but leads to confusing or unexpected results, e.g., $H_{\omega+1} = H_{\omega_1}$. So we confine our use of H_λ to cardinals only.*

For some arguments, we need to know a bound on the number of hereditary sets with a certain size.

Lemma 2.4. *Let λ be a cardinal. Then $|H_\lambda| \leq 2^{<\lambda}$.*

Proof. Let $x \in H_\lambda$, $a = \mathrm{TC}(\{x\})$ and $f : |a| \to a$ be bijective with $f(\emptyset) = x$. Let a relation $\mathcal{R} \subseteq |a|^2$ be defined by $\xi \mathcal{R} \zeta \Leftrightarrow f(\xi) \in f(\zeta)$. The pair $(|a|, \mathcal{R})$ can be encoded (transitive collapse) as a subset of $|a|^2 < \lambda$ and x is the value of $\emptyset$ in the collapse. Thus $|H(\lambda)| \leq 2^{<\lambda}$. □

Corollary 2.5. *If κ is inaccessible, then $|H_\kappa| = \kappa$.*

The following property of the Mostowski collapse will help us construct transitive models with specific properties.

Lemma 2.6. *Let $\pi : X \to M$ be the Mostowski collapse of $(X, \in)$, i.e., the function recursively defined by $\pi(x) = \{\pi(y) \mid y \in x \cap X\}$. If $a \in X$ with $\mathrm{TC}(a) \subseteq X$, then $\pi(a) = a$.*

Proof. Suppose this is false. Let a be the $\in$-minimal $a \in X$ with $\mathrm{TC}(a) \subseteq X$ and $\pi(a) \neq a$. Note that $X \cap a = a$. Let $b \in a$. Since $\mathrm{TC}(a) \subseteq X$, $b \in X$, and furthermore $\mathrm{TC}(b) \subseteq \mathrm{TC}(a) \subseteq X$. So, by the minimality of a, for all $b \in a$, $\pi(b) = b$. Hence $\pi(a) = \{\pi(b) \mid b \in a \cap X\} = \{b \mid b \in a\} = a$. □

We shall now investigate how hereditary sets interact with the technique of forcing. It turns out that this relationship is indeed quite nice. We will later require the following auxiliary lemma.

Lemma 2.7. [Wei08, 1.2] *Let $\kappa > \aleph_1$ be a cardinal, $a \in H_\kappa$ and $\varphi(z)$ be a Σ_2 formula. If $H_\kappa \models \varphi(a)$, then $\varphi(a)$ is true.*

Proof. Let ψ be a Σ_0 formula, $\varphi(z) = \exists x \forall y \psi(x, y, z)$. Let $a \in H_\kappa$ and suppose $H_\kappa \models \varphi(a)$. Assume $\forall x \exists y \neg\psi(x, y, a)$. There is $b \in H_\kappa$ such that $H_\kappa \models \forall y \psi(b, y, a)$. By our assumption, $\exists y \neg\psi(b, y, a)$ is true. Let c such that $\neg\psi(b, c, a)$. Obviously, $c \notin H_\kappa$, so let $\lambda \geq \kappa$ be the hereditary size of c. Note that $H_{\lambda^+} \models \neg\psi(b, c, a)$.

Set $\mu := |\mathrm{TC}(\{a, b\})| + \aleph_0$. $a, b \in H_\kappa$, so $\mu < \kappa$. By the Löwenheim-Skolem theorem there is some $M \prec H_{\lambda^+}$, $c \in M$, $|M| = \mu$, $\mathrm{TC}(\{a, b\}) \subseteq M$. Let $\pi : M \to N$ be the Mostowski collapsing function. By construction of M, $\pi(a) = a$ and $\pi(b) = b$. By elementarity, $M \models \neg\psi(b, c, a)$, so $N \models \neg\psi(b, \pi(c), a)$. Also, $\pi(c) \in N \subseteq H_{\mu^+} \subseteq H_\kappa$ and therefore $H_\kappa \models \neg\psi(b, \pi(c), a)$, contradicting $H_\kappa \models \forall y \psi(b, y, a)$. □

We show that hereditary size is in some natural way absolute w.r.t. forcing extensions. Because we refuse to consider H_α for non-cardinals α we always make sure that we are talking about cardinals. We start with the easy case.

Lemma 2.8. *If $\mathbb{P}$ is a forcing notion that does not collapse κ and $\dot{x} \in H_\kappa$, then for any $p \in \mathbb{P}$: $p \Vdash \dot{x} \in H_\kappa$.*

Proof. By induction on $\mathrm{rk}(\dot{x})$. The lemma holds for $\mathrm{rk}(\dot{x}) = 0$, so suppose it is true for all names with rank smaller $r = \mathrm{rk}(\dot{x})$.

Assume $\dot{x} \in H_\kappa$, write $\dot{x} = \{(\dot{y}_i, p_i) \mid i \in I\}$ for some indexing set I. By induction hypothesis, $\mathbb{1}_\mathbb{P} \Vdash \dot{y}_i \in H_\kappa$. Since $|\dot{x}| < \kappa$ and κ remains a cardinal, $\mathbb{1}_\mathbb{P} \Vdash |\dot{x}| < \kappa$. Thus (by induction) $\mathbb{1}_\mathbb{P} \Vdash \dot{x} \in H_\kappa$. □

The reversal of this result is also true, but somewhat more difficult.

Lemma 2.9. [Gol92, 3.6] *If κ is regular and $\mathbb{P} \in H_\kappa$, then for all $p \in \mathbb{P}$: If $p \Vdash \dot{x} \in H_\kappa$, there is $\ddot{x} \in H_\kappa$ with $p \Vdash \dot{x} = \ddot{x}$.*

Proof. Note that for each $x \in H_\kappa$, there is some $\lambda < \kappa$ and a sequence $(x_\alpha \mid \alpha \leq \lambda)$, $x_\alpha \in H_\kappa$ such that: For all $\alpha \leq \lambda : x_\alpha \subseteq \{x_\beta \mid \beta < \alpha\}$ and $x = x_\lambda$. Show this via induction on x. $x = \emptyset$ is obvious.

Assume this is true for all $y \in x$ and take for each $y \in x$ an appropriate $\lambda^y < \kappa$ and one such sequence $(x^y_\alpha \mid \alpha \leq \lambda^y)$. Let $\lambda = \sup_{y \in x} \lambda^y$. $\lambda < \kappa$, since $|x| < \kappa$ and κ is regular. Let $(x_\alpha)_{\alpha<\lambda}$ be the concatenation of all the $(x^y_\alpha)_{\alpha \leq \lambda^y}$ and finally set $x_\lambda = x$. This works because every $y \in x$ is at some point in the sequence.

Since $\mathbb{P} \in H_\kappa$, $\mathbb{P}$ does not collapse κ. Now let $p \in \mathbb{P}$ and $p \Vdash \dot{x} \in H_\kappa$. Then we can find names $\dot{\lambda}$, $\dot{x}_\alpha$ for the sequence discussed above. There is an ordinal $\lambda < \kappa$ such that $p \Vdash \dot{\lambda} \leq \check{\lambda}$ and since, in $V[G]$, we may set $x_\alpha = \emptyset$ for all $\dot{\lambda}^G < \alpha < \check{\lambda}^G$, we can w.l.o.g. assume that $\dot{\lambda} = \check{\lambda}$.

Now define inductively: $\ddot{x}_\alpha := \{(\ddot{x}_\beta, q) \mid \beta < \alpha \wedge q \leq p \wedge q \Vdash \dot{x}_\beta \in \dot{x}_\alpha\}$ and let $\ddot{x} = \ddot{x}_\lambda$. One can prove by induction that all $\ddot{x}_\alpha$ are in H_κ.

Now show: For all $\alpha < \lambda$, $p \Vdash \dot{x}_\alpha = \ddot{x}_\alpha$, in particular $p \Vdash \dot{x} = \ddot{x}$. Prove this via induction: Assume that for all $\beta < \alpha$, $p \Vdash \dot{x}_\beta = \ddot{x}_\beta$. Let G be $\mathbb{P}$-generic with $p \in G$. Then:

$$\begin{aligned} \ddot{x}^G_\alpha &= \left\{\ddot{x}^G_\beta \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \dot{x}_\beta \in \dot{x}_\alpha\right\} \text{ (by definition)} \\ &= \left\{\dot{x}^G_\beta \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \dot{x}_\beta \in \dot{x}_\alpha\right\} \text{ (by induction)} \\ &= \dot{x}^G_\alpha \text{ (by the Forcing Theorem).} \end{aligned}$$

For the last equality: "$\subseteq$": If there is a $q \leq p$, $q \in G$, $q \Vdash \dot{x}_\beta \in \dot{x}_\alpha$, then $\dot{x}^G_\beta \in \dot{x}^G_\alpha$. "$\supseteq$": Suppose $V[G] \models \sigma^G \in \dot{x}^G_\alpha$, then $\sigma^G = \dot{x}^G_\beta$ for some $\beta < \alpha$. Hence there is $q \leq p$, $q \in G$ that forces $\sigma = \dot{x}_\beta$. □

The following theorem is our main result in this section and shows a strong compatibility between forcing and the H-hierarchy.

Theorem 2.10. *If κ is regular and $\mathbb{P} \in H_\kappa$ then for any formula $\varphi(\bar{x})$, any $p \in \mathbb{P}$ and any names $\dot{\bar{x}} = \dot{x}_1, \ldots, \dot{x}_n$ with $p \Vdash \dot{\bar{x}} \in H_\kappa$, there are names $\ddot{\bar{x}} = \ddot{x}_1, \ldots, \ddot{x}_n \in H_\kappa$ such that:*

$$(p \Vdash H_\kappa \models \varphi(\dot{\bar{x}})) \Leftrightarrow (H_\kappa \models p \Vdash \varphi(\ddot{\bar{x}})).$$

Proof. By induction on the construction of formulae. By Lemmas 2.9 and 2.8 we may w.l.o.g. set $\ddot{\bar{x}} = \dot{\bar{x}}$. Also, the induction step for $\wedge$ is trivial.

Start with atomic formulae. W.l.o.g. let $\varphi(x, y) = x \in y$, since we can write $x = y$ equivalently as $\forall z : z \in x \leftrightarrow z \in y$ and H_κ satisfies Extensionality. Obviously, $p \Vdash$ "$H_\kappa \models \dot{x} \in \dot{y}$" iff $p \Vdash \dot{x} \in \dot{y}$. So it suffices to show $p \Vdash \dot{x} \in \dot{y} \Leftrightarrow H_\kappa \models p \Vdash \dot{x} \in \dot{y}$. Do an induction over the rank of $\dot{y}$:

If $\mathrm{rk}(\dot{y}) = 0$, $\dot{y}$ is (a name for) the empty set, so both $p \Vdash \dot{x} \in \dot{y}$ and $H_\kappa \models p \Vdash \dot{x} \in \dot{y}$ are false. Now consider $\mathrm{rk}(\dot{y}) > 0$. Suppose $p \Vdash \dot{x} \in \dot{y}$. Then $D_{\dot{x},\dot{y}} = \{r \mid \exists(\dot{z}, q) \in \dot{y} : r \leq q \wedge r \Vdash \dot{x} = \dot{z}\}$ is dense below p. We can write $D_{\dot{x},\dot{y}}$ as $\{r \mid \exists(\dot{z}, q) \in \dot{y} : r \leq q \wedge \forall \dot{a} : (r \Vdash \dot{a} \in \dot{x}) \leftrightarrow (r \Vdash \dot{a} \in \dot{z})\}$. So we can apply the inductive hypothesis and obtain $D^{H_\kappa}_{\dot{x},\dot{y}} = D_{\dot{x},\dot{y}}$ and hence $H_\kappa \models$ "$D_{\dot{x},\dot{y}}$ is dense below p". Thus $H_\kappa \models p \Vdash \dot{x} \in \dot{y}$. The backwards direction follows from Lemma 2.7.

So assume $\varphi = \neg\psi$ and that the lemma holds for ψ. For the backward direction suppose $H_\kappa \models p \Vdash \neg\psi$. If $p \Vdash \neg(H_\kappa \models \psi)$, we are done. Otherwise there is some $q \leq p$ that forces $H_\kappa \models \psi$, which by the induction hypothesis yields $H_\kappa \models q \Vdash \psi$, contradicting the assumption. The forward direction is similar.

Lastly assume $\varphi = \exists x\psi$ and that the lemma holds for ψ. Then:

$$
\begin{aligned}
& p \Vdash H_\kappa \models \exists x\psi(x) \\
\Leftrightarrow\ & \exists \dot{x} \in H_\kappa : p \Vdash H_\kappa \models \psi(\dot{x}) && \text{(by Lemmas 2.9 , 2.8, the max. principle)} \\
\Leftrightarrow\ & \exists \dot{x} \in H_\kappa : H_\kappa \models p \Vdash \psi(\dot{x}) && \text{(by induction hypothesis)} \\
\Leftrightarrow\ & H_\kappa \models \exists \dot{x} : p \Vdash \psi(\dot{x}) \\
\Leftrightarrow\ & H_\kappa \models p \Vdash \exists x\psi(x) && \text{(by the maximality principle).}
\end{aligned}
$$

□

Finally, we notice that we can do a little bit better than we actually proved here.

Remark 2.11. *By examining the proof of Lemma 2.9 we find that the condition "$\mathbb{P} \in H_\kappa$" can by replaced by "$\mathbb{P} \subseteq H_\kappa$ and satisfies the κ-cc".*

2.2 Proper forcing and CS iterations

We give an exposition of the theory of *proper forcing*, except the iteration theorem, from the ground up. The central results are the various characterizations of properness and the preservation of chain conditions along countable support iterations. We follow Jech [Jec03, Chapter 31] and explicate some additional properties implicit in his work.

We will use the following notions of closedness, unboundedness and stationarity:

Definition 2.12. [Jec03, 8.21] *Let A be an uncountable set.*

$C \subseteq [A]^\omega$ is **unbounded** *iff for every $x \in [A]^\omega$ there is a $y \in C$ with $x \subseteq y$. C is* **closed** *iff for every $\subseteq$-increasing sequence $(x_\alpha)_{\beta<\omega}$ with $x_\alpha \in C$ for all $\alpha < \omega$, $\bigcup_{\alpha<\omega} x_\alpha \in C$. $S \subseteq [A]^\omega$ is* **stationary** *iff it intersects every closed unbounded set in $[A]^\omega$.*

Remark 2.13. *All the usual concepts regarding clubsets and stationarity generalize to these notions. For details, we refer to [Jec03, Chapter 8]. We shall require the following properties.*

The appropriate version of Fodor's Lemma holds, viz., if S is stationary and $f : S \to A$ is $\in$-regressive, there is a stationary $T \subseteq S$ with $|f[T]| = 1$.

For every club $C \subseteq [C]^\omega$ there is a function $f : [A]^{<\omega} \to A$ such that $C_f = \{x \in [A]^\omega \mid \forall y \subseteq_{<\omega} x : f(y) \in x\} \subseteq C$ is a club.

If $A \subseteq B$, S is stationary in $[B]^\omega$ and T is stationary in $[A]^\omega$, then $S \restriction A = \{s \cap A \mid s \in S\}$ is stationary in $[A]^\omega$, and $T^B = \{t \in [B]^\omega \mid t \cap A \in T\}$ is stationary in $[B]^\omega$.

Now we can state the definition of a *proper forcing.*

Definition 2.14. *A notion of forcing $\mathbb{P}$ is called* **proper** *iff for all uncountable λ and all stationary $S \subseteq [\lambda]^\omega$, $\mathbb{1}_\mathbb{P} \Vdash$ "$\check{S}$ is stationary".*

As a warm-up exposition, we verify that properness is a generalization of the countable chain condition.

Lemma 2.15. *If $\mathbb{P}$ is ccc, it is proper.*

Proof. Let λ be uncountable, and let $p \in \mathbb{P}$ such that p forces that $\dot{C}$ is a name for a club in $[\lambda]^\omega$. Then there is a name $\dot{f}$ for a function $\dot{f} : \lambda^{<\omega} \to \lambda$ such that $p \Vdash C_{\dot{f}} = \{x \in [\lambda]^\omega \mid \forall y \subseteq_{<\omega} y : \dot{f}(y) \in x\} \subseteq \dot{C}$.

For each $x \in [\lambda]^\omega$ let A_x be an antichain in $\mathbb{P}$ that decides $\dot{f}(x)$. Define $g : \lambda^{<\omega} \to [\lambda]^\omega$ by $g(x) = \{\beta < \lambda \mid \exists a \in A_x : a \Vdash \dot{f}(x) = \beta\}$. Note that $g(x)$ is countable for all $x \in \operatorname{dom} g$.

Consider any $x \in C^g = \{x \in [\lambda]^\omega \mid \forall y \subseteq_{<\omega} x : g(y) \subseteq x\}$. Then, by definition of g, $p \Vdash \forall y \subseteq_{<\omega} x : \dot{f}(y) \in g(y) \subseteq x$, i.e., $p \Vdash x \in C_{\dot{f}}$. Therefore $p \Vdash C^g \subseteq \dot{C}$.

We now verify that C^g is a club and are done, since then every club in a generic extension is a superset of a club in the ground model, so stationarity is preserved. Let $x \in [\lambda]^\omega$. Construct a sequence $(x_n)_{n<\omega}$ by induction.

$x_0 = x$. Given x_n, let $x_{n+1} = x_n \cup \bigcup\{g(y) \mid y \subseteq_{<\omega} x_n\}$. Let $x^\omega = \bigcup_{n<\omega} x_n$. Clearly, $x^\omega \supseteq x$. Let $y \subseteq_{<\omega} x^\omega$, then $y \subseteq_{<\omega} x_n$ for some appropriate $n < \omega$ and so $g(y) \subseteq x_{n+1} \subseteq x^\omega$. Hence $x^\omega \in C^g$, thus C^g is unbounded.

Let $(x_n)_{n<\omega}$ be a sequence in C^g. Show that $x = \bigcup_{n<\omega} x_n \in C^g$. Just as before, let $y \subseteq_{<\omega} x$, then $y \subseteq_{<\omega} x_n$ for some appropriate $n < \omega$ and hence $g(y) \subseteq x_n \subseteq x$. Then indeed $x \in C^g$, thus C^g is closed. □

It is worth noting that the final argument in the previous proof indicates that one might also find clubs C^g for functions $g : [A]^{<\omega} \to [A]^\omega$ within any club. This is indeed true, see [Jec03, Lemma 8.26].

Now we shall investigate combinatorial (vis-à-vis the *semantic* definition) characterizations of properness.

Definition 2.16. *A condition q of some forcing $\mathbb{P}$ is called $(M, \mathbb{P})$***-generic** *iff for every maximal antichain $A \in M$, $A \cap M$ is predense below q.*

Lemma 2.17. *$\mathbb{P}$ is proper iff for every regular uncountable cardinal λ such that $\mathbb{P} \in H_\lambda$*[1] *there is a club $C \subseteq [H_\lambda]^\omega$ of countable elementary submodels $M \prec (H_\lambda, \in, <, \mathbb{P}, \dots)$ where $<$ is some fixed well-ordering of H_λ such that*

$$\forall M \in C\ \forall p \in M \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-generic.} \ (*)$$

Remark 2.18. *Henceforth, we shall simply write $M \prec H_\lambda$ when we mean $M \prec (H_\lambda, \in, <, \mathbb{P}, \dots)$. Note that the "…" mean that we can add whatever parameter we require. Also, naturally, the order of $\mathbb{P}$ shall be implicitly included. Furthermore, this means that it makes sense to talk about generic filters of $\mathbb{P}$ over M.*

Proof of the lemma. We roughly follow [Jec03, Theorem 31.7].
"⇒" Let $\mathbb{P}$ be proper and let $\lambda > 2^{|\mathbb{P}|}$ be regular. Assume that the set S of countable models $M \prec H_\lambda$ where $(*)$ fails is stationary in $[H_\lambda]^\omega$. Let $f : S \to H_\lambda$, with $f(M)$ being some $p \in M$ such that $(*)$ fails on M and p. Then f is regressive in the sense that $f(M) \in M$ for all $M \in S$. Thus by Jech's version of Fodor's Lemma [Jec03, 8.24] there is a condition $p \in \mathbb{P}$ and a stationary set $T \subseteq S$ such that for all $M \in T$, $(*)$ fails for p.

[1] We found this to be a sufficient largeness-assumption about λ. Jech [Jec03, p. 602] states this result for "sufficiently large" λ.

Let G, $p \in G$ be $\mathbb{P}$-generic. Work in $V[G]$. Every maximal antichain A (in V) below p meets G in some unique p_A. Consider the club

$$C = \{M \prec (H_\lambda)^V \mid A \in M \to p_A \in M\}.$$

Because $\mathbb{P}$ is proper, T remains stationary. So there is $M \in C \cap T$. Clearly, G is an M-generic filter, i.e., it meets all dense sets in M. Find some $q \leq p$ such that $q \Vdash$ "$\dot{G}$ is an $\check{M}$-generic filter" where $\dot{G}$ is a name for the generic filter.

We claim that q is $(M,\mathbb{P})$-generic. Assume not. Then there is some $A \in M$ such that $A \cap M$ is *not* predense below q, i.e., there is $r \leq q$ that is incompatible to all $a \in A \cap M$. But then $r \Vdash$ "$\dot{G}$ is no $\check{M}$-generic filter", because if G' is any filter with $r \in G'$, G' cannot (compatibly) intersect A in M, i.e., G' is no M-generic filter. Thus there is no such r, i.e., q is $(M,\mathbb{P})$-generic. This contradicts the choice of S.

"$\Leftarrow$" Let $\mathbb{P}$ be a forcing notion such that the lemma applies. Let λ be uncountable and $S \subseteq [\lambda]^\omega$ be stationary. Let $p \in \mathbb{P}$. Let $\dot{f}$ be a name for a function $f : [\lambda]^{<\omega} \to \lambda$. Let $\mu > \lambda$ be sufficiently large. There is a club $C \subseteq [H_\mu]^\omega$ with $\dot{f}, p \in M$ for all $M \in C$ such that $(*)$ holds for C. Because the lift $S^{H_\mu} \subseteq [H_\mu]^\omega$ is stationary, there is some $M \in C$ such that $M \cap \lambda \in S$.

Let $q \leq p$ be $(M,\mathbb{P})$-generic. We now show that q forces that $M \cap \lambda$ is closed under $\dot{f}$, i.e., the club generated by $\dot{f}$ meets S. Since every club is a superset of a club generated by such a function, this concludes the proof.

Let $y \subseteq_{<\omega} (M \cap \lambda)$. Find an antichain $A \in M$ that decides $\dot{f}(y)$. Recall that $A \cap M$ is predense below q, since q is $(M,\mathbb{P})$-generic. If now $r \leq q$ forces $\dot{f}(y) = \alpha$, then there is some $w \in A \cap M$ that is compatible with r, so, since w decides $\dot{f}(y)$, $w \Vdash \dot{f}(y) = \alpha$.

Since we can define[2] α from $w, \dot{f}$ and y, $\alpha \in M$, i.e., $\alpha \in M \cap \lambda$. Thus for all conditions $r \leq q$, $r \nVdash \dot{f}(y) \notin M \cap \lambda$, hence $q \Vdash \dot{f}(y) \in M \cap \lambda$. $\square$

We actually do not need to consider *all* these λ, as the following corollary shows.

Corollary 2.19. *Let $\mathbb{P}$ be a forcing notion. Then $\mathbb{P}$ is proper iff for* unboundedly many *regular λ with $\mathbb{P} \in H_\lambda$, there is a club $C \subseteq [H_\lambda]^\omega$ of*

[2] This is where we require that $\mathbb{P} \in H_\lambda$ and that M is an elementary submodel. The forward direction would work without these assumptions.

countable elementary submodels such that

$$\forall M \in C \; \forall p \in M \; \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-generic.}$$

Proof. The proof of Lemma 2.17 suffices. Notice that in the backwards direction we only required that μ is sufficiently large. □

Even better, instead of a proper class of cardinals, we actually only need to inspect a single μ with $\mathbb{P} \in H_\mu$.

Corollary 2.20. *Let $\mathbb{P}$ be a forcing notion and μ be a regular cardinal with $\mathbb{P} \in H_\mu$. Then $\mathbb{P}$ is proper iff there is a club $C \subseteq [H_\mu]^\omega$ of countable elementary submodels of H_μ such that*

$$\forall M \in C \; \forall p \in M \; \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-generic.} \; (*)$$

Proof. The forward direction follows directly from the work done thus far. As for the reverse direction, let μ, $\mathbb{P}$ and C as required. There is some $f : H_\mu^{<\omega} \to H_\mu$ with $C_f \subseteq C$. Consider any regular $\lambda > \mu$ with $f \in H_\lambda$. Define a club $D = \{M \in [H_\lambda]^\omega \mid M \prec H_\lambda \wedge \mathbb{P}, f \in M\}$.

We claim that D satisfies $(*)$. Let $M \in D$ and $p \in \mathbb{P} \cap M$. Since $\mathbb{P} \in H_\mu$, $p \in \mathbb{P} \cap (M \cap H_\mu)$. Since $f \in M$, $M \cap H_\mu \in C_f \subseteq C$. Hence there is an $(M \cap H_\mu)$-generic $q \leq p$.

Let $A \in M$ be a maximal antichain in $\mathbb{P}$. Since $\mathbb{P} \in H_\mu$, $A \in H_\mu$, i.e., $A \in M \cap H_\mu$. Then $A \cap M \cap H_\mu = A \cap M$ is predense below q. Thus we have verified $(*)$ on λ. □

These proofs would also show a slightly more general characterization of properness. The versions we state here are just more appropriate for our applications.

Remark 2.21. *Lemma 2.17 and its corollaries are also true if the condition $\mathbb{P} \in H_\lambda$ is weakened to $\mathbb{P} \subseteq H_\lambda$ and $\mathbb{P}$ satisfies the λ-cc.*

A useful consequence of this characterization is that sufficiently large H_λ are faithful w.r.t. their knowledge about properness. We state a general result; the bounds can be improved under nice cardinal arithmetic.

Lemma 2.22. *Let $\mathbb{P}$ be a forcing notion, λ, μ be regular cardinals with $\mathbb{P} \in H_\mu$ and $\lambda > 2^\mu$. Then: $\mathbb{P}$ is proper iff $H_\lambda \models$ "$\mathbb{P}$ is proper".*

Proof. Suppose $\mathbb{P}$ is proper. Choose a club $C \subseteq [H_\mu]^\omega$ witnessing this. Note: $|C| \leq |H_\mu| \leq 2^{<\mu} < \lambda$. Thus $C \in H_\lambda$, hence $H_\lambda \models$ "$\mathbb{P}$ is proper" by Corollary 2.20.

Suppose $H_\lambda \models$ "$\mathbb{P}$ is proper", then there is a club $C \subseteq [H_\mu]^\omega$ witnessing $\mathbb{P}$'s properness (as seen by H_λ). By the cardinality computation above, H_λ computes clubs correctly, hence C also witnesses $\mathbb{P}$'s properness in V. Thus $\mathbb{P}$ is really proper by Corollary 2.20. □

We shall frequently require that certain cardinals are preserved. The following is one of the most useful properties of proper forcing.

Lemma 2.23. *If $\mathbb{P}$ is proper, then $\mathbb{1}_\mathbb{P} \Vdash \aleph_1 = \aleph_1^V$.*

Proof. Let G be $\mathbb{P}$-generic and work in $V[G]$. Suppose ω_1^V is countable, then $\{\omega_1^V\}$ is club in $[\omega_1^V]^\omega$. But $([\omega_1]^\omega)^V$ is stationary in V, and hence in $V[G]$, i.e., $\omega_1^V \in ([\omega_1]^\omega)^V \lightning$. □

This can be extended to the following preservation lemma. The proof is a name-counting argument.

Lemma 2.24. *A proper forcing $\mathbb{P}$ of size at most 2^{ω_1} that satisfies the $\aleph_2$-cc preserves the value of 2^{ω_1}. To be more verbose, $\mathbb{1} \Vdash 2^{\omega_1} = (2^{\omega_1^V})^V$.*

Proof. Recall that properness implies that $\mathbb{1} \Vdash \omega_1 = \omega_1^V$, so we shall use ω_1 without specifying where it is computed. Suppose that G is a $\mathbb{P}$-generic filter. For each $a \subseteq \omega_1$ in $V[G]$ there is a name $\dot{a}$ in V. For each $\alpha < \omega_1$ choose a maximal antichain A_α deciding $\alpha \in \dot{a}$. Note that $|A_\alpha| < \aleph_2$. Consider the following map $f_{\dot{a}} : \omega_1 \to [\mathbb{P} \times 2]^{<\aleph_2}$:

$$\alpha \mapsto T_\alpha = \{(p,1) \mid p \in A_\alpha \wedge p \Vdash \alpha \in \dot{a}\} \cup \{(p,0) \mid p \in A_\alpha \wedge p \Vdash \alpha \notin \dot{a}\}.$$

Clearly, if $a \neq b \subseteq \omega_1$, their respective maps are different. There are at most $\left|[\mathbb{P} \times 2]^{<\aleph_2}\right|^{\omega_1} = \left|2^{\aleph_1}\right|^{\aleph_1} = 2^{\omega_1}$ such maps. Thus $\mathbb{1} \Vdash 2^{\omega_1} \leq (2^{\omega_1})^V$. □

The primary means of preserving cardinals are chain conditions. We show an appropriate result about countable support iterations.

Lemma 2.25. *Let $\kappa > \omega_1$ be regular. Let $\mathbb{P}_\kappa$ be some countable support iteration of length κ such that all stages satisfy the κ-cc. Then $\mathbb{P}_\kappa$ satisfies the κ-cc.*

Proof. Assume $A = (p_\xi \mid \xi < \kappa)$ is an antichain in $\mathbb{P}_\kappa$. W.l.o.g. assume its indexes have uncountable cofinality. Let $F(\xi) = \min\{\alpha \mid \mathrm{supp}(p_\xi) \cap \xi \subseteq \alpha\}$. F is regressive, since $\mathbb{P}_\kappa$ has countable supports. By Fodor's Lemma, e.g., [Jec03, Theorem 8.7], there is a stationary $S \subseteq \kappa$ and $\gamma < \kappa$ with $F[S] = \{\gamma\}$. Construct $\{\alpha_i \mid i \in S\} = S' \subseteq S$, $|S'| = \kappa$ with $\forall \xi < \zeta \in S' : \mathrm{supp}(p_\xi) \subseteq \zeta$ by recursion:

$$\alpha_i = \min(S \setminus (\sup_{j<i}(\mathrm{supp}(p_{\alpha_j}) \cup \alpha_j))).$$

Note that if $\xi < \zeta \in S'$, then $\mathrm{supp}(p_\xi) \subseteq \zeta$ and $\mathrm{supp}(p_\zeta) \cap \zeta \subseteq \gamma$, therefore $\mathrm{supp}(p_\xi) \cap \mathrm{supp}(p_\zeta) \subseteq \gamma$.

Since $\mathbb{P}_\gamma$ satisfies the κ-cc, there are $\xi < \zeta \in S'$ and $r' \in \mathbb{P}_\gamma$ such that $r' \leq p_\xi \restriction \gamma, p_\zeta \restriction \gamma$. Define a condition $q = (q(\alpha) \mid \alpha < \kappa) \in \mathbb{P}_\kappa$ by:

$$q(\alpha) = \begin{cases} r'(\alpha), \alpha < \gamma, \\ p_\xi(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\xi), \\ p_\zeta(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\zeta), \\ \mathbb{1}, \quad \text{otherwise.} \end{cases}$$

This is well-defined, since above γ the supports of p_ζ and p_ξ are disjoint. But then $q \leq p_\xi$ and $q \leq p_\zeta$, i.e., A is no antichain. $\lightning$ □

The following property will prove to be useful where very weak large cardinal assumptions are concerned.

Lemma 2.26. *If α is a limit ordinal, $\mathbb{P}$ is a finite or countable support forcing iteration of length α, G is $\mathbb{P}$-generic, $X \in V$, $|X| < \mathrm{cf}(\alpha)^{V[G]}$ and $S \in \mathcal{P}(X)^{V[G]}$ then there is a successor ordinal $\gamma < \alpha$ such that $S \in V[G_\gamma]$ where $G_\gamma = \{p \restriction \gamma \mid p \in G\}$.*

Proof. Note that the statement is trivial for $\mathrm{cf}(\alpha) \leq \omega$. So we may w.l.o.g. assume $\mathrm{cf}(\alpha)$ to be uncountable. Let $\dot{S}$ be a $\mathbb{P}$-name for S. For each $x \in X$ choose a $p_x \in G$ that decides $\check{x} \in \dot{S}$. Since $|X| < \mathrm{cf}(\alpha)$, there are less than $\mathrm{cf}(\alpha)$ such p_x; $\mathbb{P}$ has countable support and α is a limit, thus

$$\gamma := (\sup_{x \in X}(\mathrm{supp}(p_s))) + 1 < \alpha.$$

Notice that now, in $V[G]$, $S = \left\{x \in X \mid \exists p \in G_\gamma : p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1} \Vdash \check{x} \in \dot{S}\right\}$. We can do this computation of S in $V[G_\gamma]$. □

The final two results are the pivotal properties of proper forcing and countable support iterations we shall require for our purposes.

Lemma 2.27. *Lottery sums (Definition 3.1) of proper forcings are themselves proper.*

Proof. Let $\mathbb{P}$ be the lottery sum of $(\mathbb{Q}_\alpha \mid \alpha < \kappa)$. Let G be $\mathbb{P}$-generic. Since elements of G are pairwise compatible and if $p, q \in \mathbb{P}$, $p \in \mathbb{Q}_\alpha$, $q \in \mathbb{Q}_\beta$, $\alpha \neq \beta$, p, q are incompatible, $G \subseteq \mathbb{Q}_\alpha \cup \{\mathbb{1}\}$ for some α.

Furthermore, a set D is clearly dense in $\mathbb{P}$ if and only if $D \cap \mathbb{Q}_\alpha$ is dense in $\mathbb{Q}_\alpha$ for all $\alpha < \kappa$. Hence G is a $\mathbb{Q}_\alpha$-generic filter for some α, i.e., stationary sets are preserved between V and $V[G]$. □

Fact 2.28 (Proper Forcing Iteration Theorem)**.** *Countable support (CS) iterations of proper forcings are themselves proper. For a proof, see, e.g.,* [Jec03, Theorem 31.15].

2.3 Semiproper forcing and RCS iterations

Schlindwein [Sch93] found a simplified treatment of the *revised countable support* iterations and the corresponding iteration theorem of *semiproper* forcing. To fully understand his work we show some additional properties and results he used implicitly. Most importantly, we show a chain condition lemma. Where the main theorems are concerned, however, we just refer to Schlindwein.

Semiproper forcing is a weakening of proper forcing. In lieu of a semantic definition, we weaken the combinatorial characterization of properness.

Definition 2.29. *Let $\mathbb{P}$ be a notion of forcing, $q \in \mathbb{P}$ and M be a set. q is called $(M, \mathbb{P})$-**semigeneric** iff for every $\mathbb{P}$-name $\dot\alpha$ for a countable ordinal with $\dot\alpha \in M$, $q \Vdash \exists \beta \in M : \dot\alpha = \beta$.*

Definition 2.30. *A notion of forcing $\mathbb{P}$ is called* **semiproper** *iff for every regular uncountable cardinal λ with $\mathbb{P} \in H_\lambda$ there is a club $C \subseteq [H_\lambda]^\omega$ of countable elementary submodels $M \prec (H_\lambda, \in, <, \mathbb{P}, \dots)$, where $<$ is some fixed well-ordering of H_λ, such that*

$$\forall M \in C\ \forall p \in M \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-semigeneric.}$$

We can, however, approximate the semantic features of proper forcing on ω_1. Note that the reverse of the following lemma is not (known to be) necessarily true. A forcing notion satisfying the conclusion of that lemma is called *stationary set preserving* and may not be semiproper.

Lemma 2.31. *If* $\mathbb{P}$ *is semiproper and* $S \subseteq [\omega_1]^\omega$ *is stationary (in* V*), then* $\mathbb{1} \Vdash$ *"*$\check{S}$ *is stationary".*

Proof. Suppose that $S \subseteq [\omega_1]^\omega$ is stationary. Let $\dot{F}$ be a name for a function $F : (\omega_1^V)^{<\omega} \to \omega_1^V$ and $p \in \mathbb{P}$. We will now find some $q \leq p$ and $x \in S$ such that $q \Vdash \forall y \subseteq_{<\omega} \check{x} : \dot{F}(y) \in \check{x}$. Because any club is a superset of a club generated by such a function, $\mathbb{1}$ will then force that S is stationary. Let μ be large enough such that $\mathbb{P} \in H_\mu$ and take $C \subseteq [H_\mu]^\omega$ with

$$\forall M \in C \ \forall p \in M \ \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-semigeneric.}$$

$C' = \{M \in C \mid p, \dot{F} \in M\}$ is still club. Then there is again a club in $\{M \cap \omega_1 \mid M \in C'\}$ with respect to $[\omega_1]^\omega$, hence there is $M \in C'$ with $x := M \cap \omega_1 \in S$. Let $q \leq p$ be $(M, \mathbb{P})$-semigeneric. Now show that $q \Vdash \forall y \subseteq_{<\omega} \check{x} : \dot{F}(y) \in \check{x}$. Let $y \subseteq_{<\omega} x$. By definition of $\dot{F}$, there is a name for a countable ordinal $\dot{\alpha}$ such that $q \Vdash \dot{F}(\check{y}) = \dot{\alpha}$. $\dot{\alpha}$ is definable from $y, \mathbb{P}$ and $\dot{F}$ and hence $\dot{\alpha} \in M$. By semigenericity, $q \Vdash \dot{\alpha} \in M$ and clearly $q \Vdash \dot{\alpha} \in \omega_1^V$, so $q \Vdash \dot{\alpha} \in \check{x}$. □

This allows us to reconstruct one of the central characteristics of proper forcing.

Corollary 2.32. *If* $\mathbb{P}$ *is semiproper, then it preserves* ω_1.

Proof. Same as Lemma 2.23, using the previous lemma for stationarity. □

Just as with proper forcing, we can characterize semiproperness by examining a single cardinal.

Lemma 2.33. *Let* $\mathbb{P}$ *be a forcing notion and* μ *be regular with* $\mathbb{P} \in H_\mu$. *Then* $\mathbb{P}$ *is semiproper iff there is a club* $C \subseteq [H_\mu]^\omega$ *of elementary submodels such that*

$$\forall M \in C \ \forall p \in M \ \exists q \leq p : q \text{ is } (M, \mathbb{P})\text{-semigeneric.} \ (*)$$

Proof. "$\Rightarrow$" by definition.

For "$\Leftarrow$" suppose μ, $\mathbb{P}$ and C are as required. There is $F : H_\mu^{<\omega} \to [H_\mu]^\omega$ with $C_F = \{x \in [H_\mu]^\omega \mid \forall y \subset_{<\omega} x : F(y) \subseteq x\} \subseteq C$. First we consider any cardinal $\lambda \geq (2^{<\mu})^+ \geq |H_\mu|^+$. Then $F \in H_\lambda$.

Consider the club $D = \{M \in [H_\lambda]^\omega \mid M \prec H_\lambda,\ \mathbb{P}, F \in M\}$. Suppose $M \in D$, $p \in \mathbb{P} \cap M$ and $\dot\alpha \in M$ is a name for a countable ordinal. By Lemma 2.9, there is $\ddot\alpha \in H_{\omega_1}$ with $\mathbb{1} \Vdash \dot\alpha = \ddot\alpha$. Since $\mathbb{P} \in H_\mu$, $p, \ddot\alpha \in \mathbb{P} \cap (M \cap H_\mu)$. Furthermore, since $F \in M$, $M \cap H_\mu \in C_F \subseteq C$. By assumption, there is an $(M \cap H_\mu)$-semigeneric $q \leq p$. Thus $q \Vdash \exists \beta \in M \cap H_\mu : \ddot\alpha = \beta$, in particular $q \Vdash \exists \beta \in M : \dot\alpha = \beta$, i.e., q is $(M, \mathbb{P})$-semigeneric.

Finally let $\nu > \mu$, $\nu < \lambda$. Let $C' \subseteq [H_\lambda]^\omega$ be a club with $(*)$ and $\forall M \in C' : \nu \in C'$. Then $C = \{M \cap H_\nu \mid M \in C'\} \subseteq [H_\nu]^{<\omega}$ is a club of elementary submodels of H_ν. Let $M \in C'$, $p \in M$ and $\dot\alpha \in M \cap H_\nu$ be a name for a countable ordinal. Then there is $q \leq p$ with $q \Vdash \exists \beta \in M : \dot\alpha = \beta$; take one such β. Assume $\beta \notin M \cap H_\nu$, i.e., $\beta \notin H_\nu$, $\beta > \nu$. Then forcing with q would collapse β to a countable ordinal, but since $\beta > \mu$ and by cardinality alone, $\mathbb{P}$ satisfies the μ-cc, this cannot happen. Hence $\beta \in H_\nu$, i.e., q is $(M \cap H_\nu, \mathbb{P})$-generic. □

We again require that lottery sums of semiproper forcing are semiproper themselves. The following characterization helps with the argument.

Lemma 2.34. *$\mathbb{P}$ is semiproper iff for every $p \in \mathbb{P}$, every λ with $\mathbb{P} \in H_\lambda$ and every countable $M \prec (H_\lambda, \in, <)$ with $\mathbb{P}, p \in M$, there is a $q \leq p$ that is $(M, \mathbb{P})$-semigeneric. $<$ is here any well-ordering of H_λ.*

Proof. The reverse direction is obvious. Let $\mathbb{P}$ be semiproper, p, λ, M and μ as required and show that there is $q \leq p$ that is $(M, \mathbb{P})$-semigeneric. By definition there is some club $C \subseteq [H_\mu]^{<\omega}$ witnessing $\mathbb{P}$'s semiproperness and some function $F : H_\mu^{<\omega} \to H_\mu$ with $C_F \subseteq C$. W.l.o.g. let F be the minimal (w.r.t. $<$) such function in H_λ. By elementarity, each model $M \prec (H_\lambda, \in, <)$ knows F and thus is closed under F. Thus $M \cap H_\mu \in C$. □

Corollary 2.35. *Lottery sums (Definition 3.1) of semiproper forcing notions are themselves semiproper.*

Proof. Let $\mathbb{P}$ be the lottery sum of $(\mathbb{Q}_i \mid i < \alpha)$. Let λ be sufficiently large such that the previous lemma applies to $\mathbb{P}$ and all $\mathbb{Q}_i$. Let $p \in \mathbb{P}$. We may assume $p \neq \mathbb{1}$, since if all other conditions strengthen to a semigeneric condition, so does $\mathbb{1}$.

Then there is $i < \kappa$ with $p \in \mathbb{Q}_i$. Note that $\mathbb{Q}_i$ is computable from q and $\mathbb{P}$, since $\mathbb{Q}_i = \{r \in \mathbb{P} \mid \exists r' \neq \mathbb{1}_{\mathbb{P}} : r \geq p, r\}$. Thus for each countable $M \prec (H_\lambda, \in, <)$ with $\mathbb{P}, p \in M$ also $\mathbb{Q}_i \in M$. Now take some $(M, \mathbb{Q}_i)$-semigeneric $q \leq p$. Because forcing with q in $\mathbb{Q}_i$ amounts to the same as forcing with q in $\mathbb{P}$, q is $(M, \mathbb{P})$-semigeneric. □

Recall the following definitions/notations. They will be useful for stating the definition of revised countable supports.

Notation 2.36 ($^\frown$)**.** *Let $\alpha < \beta$. If $p : \alpha \to V$ and $q : \beta \to V$ are functions, then the* continuation *of p along q, $p^\frown q : \beta \to V$, is defined by $(p^\frown q)(i) = p(i)$ if $i < \alpha$ and $(p^\frown q) = q(i)$ otherwise.*

If $p : \alpha \to V$ and $q : \beta \setminus \alpha \to V$ are functions, then the concatenation *of p and q, $p^\frown q : \beta \to V$, is defined by $(p^\frown q)(i) = p(i)$ if $i < \alpha$ and $(p^\frown q) = q(i)$ otherwise.*

We abuse notation on $^\frown$. The two uses are always clear from the context.

Definition 2.37. *Let $(\mathbb{P}_\beta)_{\beta<\alpha}$ be some forcing iteration. The* **inverse limit** *is defined as $\tilde{\mathbb{P}}_\alpha = \{p \mid \forall \beta < \alpha : (p \restriction \beta) \in \mathbb{P}_\beta\}$; write $(\tilde{\mathbb{P}}_\alpha, \tilde{\leq})$.*

$\mathbb{P}_\alpha$ is a **direct limit** *iff for all $p \in \mathbb{P}_\alpha$ there is some $\beta < \alpha$ such that $p \restriction \beta \in \mathbb{P}_\beta$ and $\operatorname{supp}(p) \subseteq \beta$.*

Now we can state what it means for an iteration to have revised countable supports. The original definition is much more complicated than what we shall now discuss. We state a simplified approach discovered later independently by Schlindwein and Donder [Fuc92]. The basic idea seems to be that a support is not only a countable(-ish) set, but also a *name* for one. In particular, we take inverse limits not only at stages with countable cofinality, but at stages where the cofinality *will* become countable during our iteration.

Definition 2.38. *A condition $p \in \tilde{\mathbb{P}}_\alpha$ is said to have* **revised countable support** *if:*

$$\forall q \tilde{\leq} p \ \exists \beta < \alpha \exists r \leq q \restriction \beta : r \Vdash \text{"}\operatorname{cof}(\alpha) = \omega \vee \operatorname{supp}(p \restriction [\beta, \alpha)) = \emptyset\text{" (R).}$$

The **revised countable limit** *of the iteration is*

$$\mathrm{Rlim} = \{p \in \tilde{\mathbb{P}}_\alpha \mid p \text{ has revised countable support}\}.$$

An iteration has **revised countable supports (RCS)** *if each limit stage is the revised countable limit.*

The following result is part of a historic effort by Foreman, Magidor and Shelah [FMS88] to show the consistency of MM relative a supercompact. We shall make no effort to reproduce a proof here. However, Schlindwein's treatment is quite approachable.

Fact 2.39 (Central Theorem of Semiproper Forcing)**.** *RCS iterations are indeed forcing iterations and an RCS iteration of semiproper forcings is semiproper itself. For a proof see [Sch93]. Some fragments of these properties will be shown below.*

We collect some additional properties of RCS iterations. The following three results are quite useful technicalities.

Lemma 2.40. *Let $\mathbb{P}_\alpha$ be an RCS iteration. If $p \in \mathbb{P}_\alpha$ and $q \tilde{\leq} p$, then the $\beta < \alpha$ witnessing this in (R) can be chosen arbitrarily large.*

Proof. Do an induction on α. Let $p \in \mathbb{P}_\alpha$, q, β, r as in (R) and $\beta < \gamma < \alpha$. Let $r' = r^\frown(q \restriction \gamma)$.

Show by induction on $\delta \leq \gamma$ that $(r' \restriction \delta) \in \mathbb{P}_\delta$. $\delta \leq \beta$ is obvious. Successor steps are also clear, so let δ be a limit with $\beta < \delta \leq \gamma$. By the induction hypothesis (on δ), for all $\delta' < \delta$, $(r' \restriction \delta') \in \mathbb{P}_{\delta'}$, so $(r' \restriction \delta) \in \tilde{\mathbb{P}}_\delta$. Let $q' \tilde{\leq} (r' \restriction \delta)$. Note that $(q \restriction \delta) \in \mathbb{P}_\delta$ and by construction $(r' \restriction \delta) \tilde{\leq} (q \restriction \delta)$, so $q' \tilde{\leq} (q \restriction \delta)$. Hence there is $\beta' < \delta$ and $r'' \leq (q' \restriction \beta')$ such that

$$r'' \Vdash \text{"} \operatorname{cof}(\delta) = \omega \vee \operatorname{supp}(q \restriction [\beta', \delta)) = \emptyset\text{"}.$$

By the induction hypothesis (on α) we can choose $\beta' > \beta$. So since above β, $r = q$ this means:

$$r'' \Vdash \text{"} \operatorname{cof}(\delta) = \omega \vee \operatorname{supp}(r \restriction [\beta', \delta)) = \emptyset\text{"}.$$

Hence $(r' \restriction \delta) \in \mathbb{P}_\delta$. This finishes the induction (on δ), so $r' = (r' \restriction \gamma) \in$

$\mathbb{P}_\gamma$. Then $r' \leq (q \restriction \gamma)$ and $r' \restriction \beta \leq r$, i.e.,

$$r' \Vdash \text{"}\operatorname{cof}(\alpha) = \omega \vee \operatorname{supp}(p \restriction [\beta, \alpha)) = \emptyset\text{"}.$$

Thus q, γ, r' satisfy (R), because $\gamma > \beta$, i.e., $[\gamma, \alpha) \subseteq [\beta, \alpha)$. □

Corollary 2.41. *Let $\mathbb{P}_\alpha$ be an RCS iteration and $\gamma \leq \alpha$. If $p \in \mathbb{P}_\gamma$ and $p' \in \mathbb{P}_\alpha$ such that $p \leq (p' \restriction \gamma)$, then $(p^\frown p') \in \mathbb{P}_\alpha$.*

Proof. By induction. $\alpha = 0$ and successor steps are trivial. So suppose α is a limit. Consider any $\beta < \alpha$ and do a distinction of cases: If $\gamma \leq \beta < \alpha$, then $(p^\frown p') \restriction \beta = (p^\frown (p' \restriction \beta)) \in \mathbb{P}_\beta$ by the induction hypothesis. If $\beta \leq \gamma$, then trivially $(p^\frown p') \restriction \beta = p \restriction \beta \in \mathbb{P}_\beta$.

Hence $(p^\frown p') \in \tilde{\mathbb{P}}_\alpha$. Now verify (R) for $(p^\frown p')$: Let $q \tilde{\leq} (p^\frown p')$. Then, since $p \leq (p' \restriction \gamma)$, $q \tilde{\leq} p'$. So, since $p' \in \mathbb{P}_\alpha$, there is $\beta < \alpha$, $r \leq q \restriction \beta$ such that $r \Vdash \text{"}\operatorname{cof}(\alpha) = \omega \vee \operatorname{supp}(p' \restriction [\beta, \alpha)) = \emptyset\text{"}$. We can choose $\beta > \gamma$ by the previous lemma. Then $\operatorname{supp}(p' \restriction [\beta, \alpha)) = \operatorname{supp}(p^\frown p' \restriction [\beta, \alpha))$ and we are done. □

Corollary 2.42. *In particular, if $p \in \mathbb{P}_\gamma$, then $p^\frown \mathbb{1}^\frown \ldots {}^\frown \mathbb{1} \in \mathbb{P}_\alpha$.*

The next observation will be useful in our applications.

Lemma 2.43. *If $\mathbb{P}_\kappa$ is an RCS iteration of uncountable regular length κ that satisfies the κ-cc, then $\mathbb{P}_\kappa$ is a direct limit.*

Proof. If this is false, there is $p \in \mathbb{P}_\kappa$ with cofinal support. Then there would be some $\beta < \kappa$, $r \leq p \restriction \beta$ such that $r \Vdash \operatorname{cof} \kappa = \omega$, i.e., $r^\frown \mathbb{1}_\kappa \Vdash \operatorname{cof} \kappa = \omega$. This contradicts the κ-cc. □

We want to verify some kind of chain condition lemma on RCS iterations. Unfortunately, the technicalities involved are quite unpleasant; which is why we have put the most tedious part in the following lemma.

Lemma 2.44. *Let $\mathbb{P}_\alpha$ be an RCS iteration. Let $\xi \leq \alpha$ and $p, q \in \mathbb{P}_\alpha$ be incompatible with $\operatorname{supp}(p) \cap \operatorname{supp}(q) \subseteq \xi$. Then $p \restriction \xi$ and $q \restriction \xi$ are incompatible.*

Proof. Assume there is some $r' \leq (p \restriction \xi), (q \restriction \xi)$. Find some $r : \alpha \to V$ with:

$$\begin{aligned} &r \restriction \xi = r', \\ &r \restriction (\mathrm{supp}(p) \setminus \xi) = p \restriction (\mathrm{supp}(p) \setminus \xi), \\ &r \restriction (\mathrm{supp}(q) \setminus \xi) = q \restriction (\mathrm{supp}(q) \setminus \xi), \\ &r(i) = \mathbb{1}_i \text{ if } i \notin \mathrm{supp}(r') \cup \mathrm{supp}(p) \cup \mathrm{supp}(q). \end{aligned}$$

This is possible, since above ξ the supports of p and q are disjoint.

Now show that $r \in \mathbb{P}_\alpha$. To this end, show by induction on $\delta \leq \alpha$ that $(r \restriction \delta) \in \mathbb{P}_\delta$. $\delta \leq \xi$ and successor steps are trivial, so assume $\delta > \xi$ is a limit. By the induction hypothesis, $(r \restriction \delta) \in \tilde{\mathbb{P}}_\delta$.

Let $r_1 \tilde{\leq} (r \restriction \delta)$, in particular this means $r_1 \tilde{\leq} (p \restriction \delta)$. By Lemma 2.40 there is $\beta_1 < \delta$, $\xi < \beta_1$ and $s_1 \leq (r_1 \restriction \beta_1)$ such that

$$s_1 \Vdash \text{" } \mathrm{cof}(\delta) = \omega \vee \mathrm{supp}(p \restriction [\beta_1, \delta)) = \emptyset\text{"}.$$

By construction $s_1 \leq (r_1 \restriction \beta_1) \leq (r \restriction \beta_1) \leq (q \restriction \beta_1)$, therefore we find $r_2 = (s_1 {}^\frown (q \restriction \delta)) \in \mathbb{P}_\delta$. Then $r_2 \tilde{\leq} (q \restriction \delta)$, so there are $\beta_2 < \delta$, $\beta_1 < \beta_2$ and $s_2 \leq (r_2 \restriction \beta_2)$ such that

$$s_2 \Vdash \text{" } \mathrm{cof}(\delta) = \omega \vee \mathrm{supp}(q \restriction [\beta_2, \delta)) = \emptyset\text{"}.$$

Then $s_2 \leq s_1 {}^\frown \mathbb{1}_{\beta_2}$. So, since $\mathrm{supp}(r) \setminus (\mathrm{supp}(p) \cup \mathrm{supp}(q)) \subseteq \xi < \beta_1 < \beta_2$:

$$s_2 \Vdash \text{" } \mathrm{cof}(\delta) = \omega \vee \mathrm{supp}(r \restriction [\beta_2, \delta)) = \emptyset\text{"}.$$

So $r \in \mathbb{P}_\delta$. This finishes the induction on δ. So $r = (r \restriction \delta) \in \mathbb{P}_\delta$, i.e., p and q are compatible. □

The following consequence of Lemma 2.32 would hold in more general situations.

Lemma 2.45. *If $\mathbb{P}$ is semiproper and α is an ordinal with cofinality ω_1, then $\mathbb{1}_\mathbb{P} \Vdash \mathrm{cof}\, \alpha = \omega_1$.*

Proof. Suppose there is $p \in \mathbb{P}$ with $p \Vdash \mathrm{cof}\, \alpha \leq \omega$. Let $(\beta_i)_{i<\omega_1}$ be cofinal in α. Consider some $\mathbb{P}$-generic G, $p \in G$, and work in $V[G]$. Let $(\gamma_n)_{n<\omega}$ be a cofinal sequence in α.

Define a sequence $(i_n)_{n<\omega}$: $i_n = \min\{i \in \omega_1 \mid \beta_i > \gamma_n\}$. We claim that $\sup_{n<\omega} i_n = \omega_1$. If not, the γ_n would be bounded by some β_i, i.e., (γ_n) could not be cofinal in α. Thus ω_1^V has cofinality ω as witnessed by $(i_n)_{n<\omega}$, i.e., it is collapsed. This contradicts Lemma 2.32. □

We are not sure if the following is the best possible, i.e., with the least required assumptions, version of a chain condition result on RCS iterations. In particular, we would like to remove the condition that the stages are semiproper (though for purely aesthetic reasons), but were not able to do so. Nevertheless, it is quite sufficient for our applications. As we have already dealt with the problems introduced by RCS iterations, the proof is basically the argument of Lemma 2.25.

Lemma 2.46. *Let $\kappa > \omega_1$ be regular and let $\mathbb{P}_\kappa$ be a revised countable support iteration of length κ such that all stages are semiproper and satisfy the κ-cc. Then $\mathbb{P}_\kappa$ satisfies the κ-cc.*

Proof. First notice that $\mathbb{P}_\kappa$ is a direct limit. If not, there is $p \in \mathbb{P}$ that is not constantly $\mathbb{1}$ above some point. Then there would be some $\beta < \kappa$, $r \leq p \restriction \beta$ such that $r \Vdash \operatorname{cof} \kappa = \omega$. But $\mathbb{P}_\beta$ satisfies the κ-cc, so it cannot collapse the cofinality of κ.

Let $T \subseteq \kappa$ be the stationary set of all ordinals with cofinality ω_1. Consider any $\alpha \in T$. By the previous lemma no stage can collapse α's cofinality, so $\mathbb{P}_\alpha$ must be a direct limit.

Now let $A = (p_\xi \mid \xi < \kappa)$ an antichain in $\mathbb{P}_\kappa$ of size κ, w.l.o.g. indexed by ordinals in T. Let $F(\xi) = \min\{\alpha \mid \operatorname{supp}(p_\xi) \cap \xi \subseteq \alpha\}$. F is regressive by construction of T, because if $\mathbb{P}_\xi$ is a direct limit, $\operatorname{supp}(p) \cap \xi < \xi$ for any p. By Fodor's Lemma [Jec03, Theorem 8.7], there is a stationary $S \subseteq T$ and $\gamma < \kappa$ with $F[S] = \{\gamma\}$.

We recursively construct a set $\{\alpha_i \mid i \in S\} = S' \subseteq S$, $|S'| = \kappa$ with $\forall \xi < \zeta \in S' : \operatorname{supp}(p_\xi) \subseteq \zeta$:

$$\alpha_i = \min(S \setminus (\sup_{j<i}(\operatorname{supp}(p_{\alpha_j}) \cup \alpha_j))).$$

This works because $\mathbb{P}_\kappa$ is a direct limit, i.e., all the supports are bounded. Note that if $\xi < \zeta \in S'$, then $\operatorname{supp}(p_\xi) \subseteq \zeta$ and $\operatorname{supp}(p_\zeta) \cap \zeta \subseteq \gamma$, therefore $\operatorname{supp}(p_\xi) \cap \operatorname{supp}(p_\zeta) \subseteq \gamma$. Then we are done by Lemma 2.44. □

Some applications require the following analogue to Lemma 2.26.

Lemma 2.47. *If κ is a regular cardinal, $\mathbb{P}$ is an RCS iteration of length κ that satisfies the κ-cc, G is $\mathbb{P}$-generic, $X \in V$, $|X| < \kappa$ and $S \in \mathcal{P}(X)^{V[G]}$ then there is some $\gamma < \kappa$ such that $S \in V[G_\gamma]$ where $G_\gamma = \{p \restriction \gamma \mid p \in G\}$.*

Proof. Note that if $\mathbb{P}$ has the κ-cc, no condition – and no condition in any stage – may collapse the cofinality of κ and thus $\mathbb{P}$ must be a direct limit.

Let $\dot{S}$ be a $\mathbb{P}$-name for S. For each $x \in X$ choose a $p_x \in G$ that decides $\check{x} \in \dot{S}$. Since $|X| < \kappa$, there are less than $\kappa = \operatorname{cof} \kappa$ such p_x; $\mathbb{P}$ is a direct limit, thus $\gamma := (\sup_{x \in X}(\operatorname{supp}(p_s))) + 1 < \kappa$.

Notice that now, in $V[G]$, $S = \left\{x \in X \mid \exists p \in G_\gamma : p^\frown \mathbb{1}^\frown \dots ^\frown \mathbb{1} \Vdash \check{x} \in \dot{S}\right\}$. We can do this computation of S in $V[G_\gamma]$. □

We will frequently apply the Factor Lemma. We should mention that it actually applies to RCS iterations.

Remark 2.48 (Factor Lemma)**.** *The applications usually center around the Factor Lemma applied to some iteration. An appropriate version, viz., RCS iterations factor into RCS iterations, can be found in [Sch93, Theorem 5].*

3 The LHMC Iterations

3.1 Definitions and basic properties

In this section we introduce the central technique used in this thesis and some ways to modify it. We proceed to investigate some niceness properties of these forcing iterations. The basic idea is simple: We force with (in some sense) all small, minimal counterexamples to some forcing axiom, and do the same in each stage of an iteration. We then hope that no counterexamples are left if we do this often enough; naturally, "often enough" means of large cardinal length. The first definition captures what we mean by forcing with "all" counterexamples.

Definition 3.1. *Let* $\{\mathbb{P}_\alpha, \alpha < \lambda\}$ *be a set of forcing notions. The* **lottery sum** *of the* $\mathbb{P}_\alpha$ *is their disjoint union* $\mathbb{P}$ *with a new* $\mathbb{1}$ *such that* $\mathbb{1} > p$ *for all* $p \in P_\alpha$*,* $\alpha < \lambda$*.*

We can now define a general scheme to generate the iterations we will use. For ease of notation, we make a distinction between the proper and the semiproper case.

Definition 3.2. *Let* $\mathcal{A}$ *be a forcing axiom, i.e., a statement of the form "for all forcing notions* $\mathbb{P}$*,* $\varphi(\mathbb{P})$*" for some statement* φ*. Let* κ *be some ordinal.*

- *The* **iterated lottery sum of hereditarily minimal counterexamples (LHMC iteration) of** $\mathcal{A}$ **with length** κ *is the countable support iteration of* $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa)$*, where* $\mathbb{P}_\alpha$ *and* $\dot{\mathbb{Q}}_\alpha$ *are defined by induction: Let* $\dot{\mathbb{Q}}_\alpha$ *be a hereditarily minimal* $\mathbb{P}_\alpha$*-name for the lottery sum of all* proper *counterexamples to* $\mathcal{A}$ *of minimal hereditary size smaller* κ*.*

- *The* **revised LHMC iteration of** $\mathcal{A}$ **with length** κ *is the RCS iteration of* $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa)$*, where* $\mathbb{P}_\alpha$ *and* $\dot{\mathbb{Q}}_\alpha$ *are defined by induction: Let* $\dot{\mathbb{Q}}_\alpha$ *be a hereditarily minimal* $\mathbb{P}_\alpha$*-name for the lottery sum of all* semiproper *counterexamples to* $\mathcal{A}$ *of minimal hereditary size smaller* κ*.*

Note that if at some point $\dot{\mathbb{Q}}_\alpha$ is a name for the trivial forcing $\{\mathbb{1}\}$, all $\dot{\mathbb{Q}}_\beta$, $\beta \geq \alpha$ will be names for the trivial forcing. Say that the iteration *stops* if this happens. Unfortunately we were unable to find any interesting results related to this.

Question 3.3. *Are there any interesting effects related to a LHMC iteration stopping?*

Some tentative approaches towards an analysis of stopping LHMC iterations are in the applications.

Furthermore, given a forcing axiom not about (semi)proper forcings, one would want to lift the restriction to only (semi)proper counterexamples. Since we will not consider such axioms in this thesis we found the definition as it is more convenient to use. An interested reader should note that most properties we show would directly generalize to such a broader definition. However, one should be cautious whenever the size of the continuum is concerned.

In the introduction we briefly discussed the *proper lottery preparation.* It is worth observing that we are doing at least some things differently.

Remark 3.4. *LHMC iterations are similar to, but not necessarily forcing equivalent to the lottery preparation by Hamkins and Johnstone, defined in [HJ09].*

Proof. A LHMC iteration may completely leave out some forcings that are never minimal counterexamples, whereas a lottery preparation will use them in some lottery sum. A simple example:

If κ is H_{κ^+}-reflecting, the proper lottery preparation of κ would collapse κ to $\aleph_2$ [HJ09, Theorem 2]. But if $\mathcal{A}$ holds in the ground model, the respective LHMC iteration is the trivial forcing and won't collapse κ. □

It is entirely possible – and reasonable – to make the LHMC iterations more similar to the proper lottery preparation. Recall Definition 1.1 from the introduction: the proper lottery preparation forces with *all* proper forcings in some H_λ.

Remark 3.5. *Our definition could be varied by defining each stage as* all *(semi)proper forcings with the hereditary size of a hereditarily minimal counterexample to $\mathcal{A}$ (instead of just the counterexamples of that size). However, the previous remark would still apply.*

The observation in Remark 3.4 sometimes does more harm than good, since we may want to firmly control certain cardinalities. To rectify the situation, we introduce a way to modify the LHMC iterations. We investigate these modifications in the following section.

Definition 3.6. *LHMC iterations can be varied as follows:*

- *A LHMC iteration is called* **adding** *iff in the induction step at cofinally many α, $\dot{\mathbb{Q}}_\alpha$ is replaced by $\dot{\mathbb{Q}}_\alpha * \mathrm{Fn}(\omega, 2)$. Note that if $\dot{\mathbb{Q}}_\alpha$ is (semi)proper, this is still a (semi)proper forcing.*

- *A LHMC iteration is called* **collapsing** *iff in the induction step at a cardinal, $\dot{\mathbb{Q}}_\alpha$ is replaced by $\dot{\mathbb{Q}}_\alpha * \mathrm{Col}(\omega_1, |\alpha|)$. Note that if $\dot{\mathbb{Q}}_\alpha$ is (semi)proper, this is still a (semi)proper forcing.*

Remark 3.7. *Usually, the applications allow to add collapsing or adding without changing the proofs at all.*

The following results show that LHMC iterations of large cardinal length are relatively well-behaved. In particular this includes a number of "smallness" conditions, e.g., hereditary size and chain conditions. First we require an auxiliary lemma.

Lemma 3.8. *Let $\mathbb{P}$ be a notion of forcing and δ be a cardinal. If $|\mathbb{P}| < \delta$, then $\mathbb{1} \Vdash 2^\delta = (2^\delta)^V$.*

Proof. This is the proof of Lemma 2.24. Suppose that G is a $\mathbb{P}$-generic filter. For each $a \subseteq \delta$ in $V[G]$ there is a name $\dot{a}$ in V. For each $\alpha < \delta$ choose a maximal antichain A_α deciding $\alpha \in \dot{a}$. Note that $|A_\alpha| < \delta$. Consider the following map $f_{\dot{a}} : \delta \to [\mathbb{P} \times 2]^{<\delta}$:

$$\alpha \mapsto T_\alpha = \{(p, 1) \mid p \in A_\alpha \wedge p \Vdash \alpha \in \dot{a}\} \cup \{(p, 0) \mid p \in A_\alpha \wedge p \Vdash \alpha \notin \dot{a}\}.$$

Clearly, if $a \neq b \subseteq \delta$, their respective maps are different. Thus there are at most $\left|[\mathbb{P} \times 2]^{<\delta}\right|^\delta \leq 2^\delta$ many such maps. Therefore $\mathbb{1} \Vdash 2^\delta \leq (2^\delta)^V$. □

The following theorem is the central "smallness" property of LHMC iterations.

Theorem 3.9. *Let $\mathbb{P}_\kappa$ be some (revised, adding, collapsing) LHMC iteration of length κ. If κ is inaccessible and $\alpha < \kappa$, then $|\mathbb{P}_\alpha| < \kappa$.*

Proof. By induction on α: If $\alpha = 0$, $\mathbb{P}_\alpha$ is a union of forcing notions with hereditary size $\gamma < \kappa$, so $\mathbb{P}_\alpha \subseteq H_{\gamma^+}$. Therefore $|\mathbb{P}_\alpha| \leq \left|H_{\gamma^+}\right| \leq 2^\gamma < \kappa$.

If $\alpha = \beta+1$, $\mathbb{P}_\beta$ forces that $\mathbb{P}_\alpha$ is a union of forcing notions with hereditary size $\gamma < \kappa$, so exactly as above, $\mathbb{1}_\beta \Vdash |\mathbb{Q}_\alpha| \leq \left|H_{\gamma+}\right| \leq 2^\gamma$. Now, since

$|\mathbb{P}_\beta| < \kappa$, there is some $\delta > \max\{\gamma, |\mathbb{P}_\beta|\}$, $\delta < \kappa$, i.e., $\mathbb{1}_\beta \Vdash 2^\gamma \leq 2^\delta = (2^\delta)^V$. Thus, since κ is inaccessible, $|\mathbb{P}_\alpha| \leq 2^\delta < \kappa$.

Suppose $\gamma < \kappa$ is a limit and for all $\alpha < \gamma$, $|\mathbb{P}_\alpha| < \kappa$. Since κ is regular, there is some λ such that for all $\alpha < \gamma$, $\lambda > |\mathbb{P}_\alpha|$. Notice that $|\mathbb{P}_\gamma| \leq \Pi_{\alpha<\gamma} |\mathbb{P}_\alpha|$, since $p \mapsto (p \restriction \alpha)_{\alpha<\gamma}$ is injective. Thus we conclude $\Pi_{\alpha<\gamma} |\mathbb{P}_\alpha| \leq \Pi_{\alpha<\gamma}\lambda = \lambda^\gamma < \kappa$. □

Now we can quickly infer further nice things about our iterations. We shall use this opportunity to consolidate some similar results on proper and semiproper forcing.

Corollary 3.10. *Let $\mathbb{P}_\kappa$ be a (revised, adding, collapsing) LHMC iteration. If κ is inaccessible and $\alpha < \kappa$, then $\mathbb{P}_\alpha \in H_\kappa$.*

Proof. By induction. $\alpha = 0$ and limit steps are clear. Let $\alpha = \beta + 1$. The proof of Theorem 3.9 shows that $\mathbb{1}_\beta \Vdash \dot{\mathbb{Q}}_\beta \in H_\kappa$. Since $\mathbb{Q}_\beta$ is chosen hereditarily minimal and by Lemma 2.9 there is a name for $\mathbb{Q}_\beta$ in H_κ, we know $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta \in H_\kappa$. □

Corollary 3.11. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration, then it satisfies the κ-cc by Lemma 2.25 resp. Lemma 2.46.*

Corollary 3.12. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration, then $\mathbb{P}_\kappa$ is a direct limit.*

Proof. This is clear for countable support and implied by Lemma 2.43 for RCS. □

Corollary 3.13. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration, it has at most cardinality κ.*

Proof. $\mathbb{P}_\kappa$ is a direct limit, so for each $p \in \mathbb{P}_\kappa$, there is $\beta_p < \kappa$ such that for all $\alpha > \beta_p$, $p(\alpha) = \mathbb{1}$. For each $\beta < \kappa$, let $Q_\beta = \{p \in \mathbb{P}_\kappa \mid \beta_p = \beta\}$. Note that $p \mapsto p \restriction \beta$ is an injection $Q_\beta \to \mathbb{P}_\beta$ and hence all Q_β have cardinality less than κ. Clearly $\mathbb{P}_\kappa = \bigcup_{\beta<\kappa} Q_\beta$, thus $|\mathbb{P}_\kappa| \leq \kappa$. □

Corollary 3.14. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration, $\mathbb{P}_\kappa \in H_{\kappa^+}$, i.e., $|\mathrm{TC}(\mathbb{P}_\kappa)| = \kappa$.*

Proof. Let $p \in \mathbb{P}_\kappa$. For each $\alpha < \kappa$, $p \restriction (\alpha + 1) \in H_\kappa$, so in particular $(\alpha, p(\alpha)) \in H_\kappa$, i.e., for all $t \in p$, $t \in H_\kappa$. Then $\mathrm{TC}(p) = p \cup \bigcup_{t\in p} \mathrm{TC}(t)$ has cardinality κ. Thus $\mathrm{TC}(\mathbb{P}_\kappa) = \mathbb{P}_\kappa \cup \bigcup_{p\in\mathbb{P}} \mathrm{TC}(p)$ has size κ. □

Sometimes we want an iteration to be even smaller. Luckily, Corollary 3.12 allows us to cut short the conditions in a LHMC iteration. So we do not have to repeat this argument, we shall just reference the following remark whenever required.

Remark 3.15. *If κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration, we may w.l.o.g. assume $\mathbb{P}_\kappa \subseteq H_\kappa$.*

Proof. $\mathbb{P}_\kappa$ is a direct limit by Corollary 3.12. So for each $p \in \mathbb{P}_\kappa$ we may write $p = p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1}$, i.e., we can "forget" the trailing $\mathbb{1}$s and reappend them implicitly or once needed. Thus we can say $p \in H_\kappa$ by Corollary 3.10. $\square$

This leads to the question whether a LHMC iteration could be even definable inside H_κ. The proof is a little messy, but generalizes whenever required.

Lemma 3.16. *Let κ be inaccessible. Then we can define the (revised, adding, collapsing) LHMC to* PFA *iteration $\mathbb{P}_\kappa$ in H_κ.*

Proof. By the previous Remark, it suffices to define $\mathbb{P}_\kappa$ as a class sequence $(\mathbb{P}_\alpha \mid \alpha < \kappa)$ where the $\mathbb{P}_\alpha$ are the initial segments of $\mathbb{P}_\kappa$. We shall give a recursive definition of that sequence. Suppose γ is a limit and we have defined $\mathbb{P}_\alpha$ for $\alpha < \gamma$. Then we can define $\mathbb{P}_\gamma$ as a (revised) countable support limit.

Now let $\alpha = \beta + 1$ and let $\mathbb{P}_\beta$ be defined. We can define $\varphi(\dot{Q}) =$"$\dot{Q}$ is a hereditarily minimal $\mathbb{P}_\beta$-name for the lottery sum of all proper counterexamples to PFA of minimal hereditary size". Find such $\dot{Q}$ and let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{Q}$. We now need to show that $\varphi(\dot{Q})$ holds in V.

We argue that it is sufficient to see that V also believes that $\dot{Q}$ is a name for a lottery sum of *proper* forcings. Note that $\dot{Q}$ is indeed a name for a lottery sum consisting of forcings with hereditary size *smaller* κ by Lemma 2.8. All other properties except properness in φ are easily absolute between H_κ and V because by Lemma 2.9 $V[G]$ and $H_\kappa[G]$ agree on the relevant witnesses.

We shall now deal with $\dot{Q}$'s properness in a generic extension. We have assumed that $H_\kappa \models$ "$\mathbb{1}_\beta \Vdash \dot{Q}$ is a proper lottery sum". By Theorem 2.10, $\mathbb{1}_\beta \Vdash$ "$H_\kappa \models \dot{Q}$ is a proper lottery sum". As in the proof of Theorem 3.9 we know that there is some $\delta < \kappa$ such that $\mathbb{1}_\beta \Vdash \dot{Q} \in H_{2^\delta} \wedge 2^\delta < \kappa$. W.l.o.g.

let 2^δ be regular or find some regular $\lambda > 2^\delta$, $\lambda < \kappa$. Then we can apply Lemma 2.22 in a generic extension to see that $\dot{Q}$ is indeed a name for a proper forcing. □

3.2 The size of the continuum

Related techniques, using fast functions (e.g., Laver functions) on κ, would usually collapse $\kappa = \aleph_2 = \mathfrak{c}$, just due to the fact that the iterations are broad enough for the appropriate collapsing and Cohen forcings to appear *somewhere*. However, these forcings may never be minimal counterexamples to whatever axiom we are considering, so the size of the continuum is more open with our approach. We can however easily reproduce the classical results by making the LHMC iteration adding and collapsing.

We start with another name-counting argument.

Lemma 3.17. *Suppose κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration of length κ. Then $\mathbb{1} \Vdash 2^{\omega_1} \leq \kappa$. Note that we would want to specify the ω_1 of a generic extension here. But ω_1 is preserved anyway.*

Proof. Suppose that G is a $\mathbb{P}_\kappa$-generic filter. For each $a \subseteq \omega_1$ in $V[G]$ there is a name $\dot{a}$ in V. Because ω_1 is preserved, we can do the following: For each $\alpha < \omega_1$ choose a maximal antichain A_α deciding $\alpha \in \dot{a}$. Recall that $|A_\alpha| < \kappa$ by Corollary 3.11. Consider the following map $f_{\dot{a}} : \omega_1 \to [\mathbb{P}_\kappa \times 2]^{<\kappa}$:

$$\alpha \mapsto T_\alpha = \{(p, 1) \mid p \in A_\alpha \wedge p \Vdash \alpha \in \dot{a}\} \cup \{(p, 0) \mid p \in A_\alpha \wedge p \Vdash \alpha \notin \dot{a}\}.$$

Clearly, if $a \neq b \subseteq \omega_1$, their respective maps are different. Note that since κ is inaccessible (so the cardinal arithmetic works out, viz., $\kappa^{<\kappa} = \kappa$ and $\omega_1 < \kappa$), there are at most $|[\mathbb{P}_\kappa \times 2]^{<\kappa}|^{\omega_1} \leq |\kappa^{<\kappa}|^{\omega_1} = |\kappa^{\omega_1}| = \kappa$ many such maps. Therefore $\mathbb{1}_\kappa \Vdash 2^{\omega_1} \leq \kappa$. □

Now we can finally investigate how LHMC iterations affect the size of the continuum.

Corollary 3.18. *Suppose κ is inaccessible and $\mathbb{P}_\kappa$ is a (revised, adding, collapsing) LHMC iteration of length κ. Then $\mathbb{1} \Vdash \mathfrak{c} \leq \kappa$.*

Proof. Surely $\mathbb{1} \Vdash \mathfrak{c} \leq 2^{\omega_1} \leq \kappa$ by the previous lemma. □

Corollary 3.19. *Suppose κ is an inaccessible cardinal and $\mathbb{P}_\kappa$ is an* adding *(revised, collapsing) LHMC iteration of length κ. Then $\mathbb{1} \Vdash \mathfrak{c} = \kappa = 2^{\omega_1}$.*

Proof. Since Cohen reals are added cofinally often, $\mathbb{1} \Vdash \mathfrak{c} \geq \kappa$. And always $2^{\omega_1} \geq \mathfrak{c}$. □

Lemma 3.20. *Suppose κ is an inaccessible cardinal and $\mathbb{P}_\kappa$ is a (semi)proper* collapsing *(adding, revised) LHMC iteration of length κ. Then $\mathbb{1} \Vdash \kappa = \aleph_2$.*

Proof. $\mathbb{P}_\kappa$ satisfies the κ-cc and all cardinals λ, $\omega_1 < \lambda < \kappa$ will be collapsed. Hence κ will become $\aleph_2$. □

Corollary 3.21. *If $\mathbb{P}_\kappa$ is a (semi)proper* adding and collapsing *LHMC iteration of inaccessible length, $\mathbb{1} \Vdash \mathfrak{c} = \aleph_2 = \kappa = 2^{\omega_1}$. This is usually the desired behavior.*

3.3 Special counterexamples

We recall a general notion of *bounded* fragments of the Proper Forcing Axiom. The simplest instance of such fragments, the BPFA, was prominently considered by Goldstern and Shelah in [GS95]. These axioms apply to forcings of arbitrary cardinality, but restrict the size of the sets we search genericity for. We introduce a notion (*special counterexamples*) to code sufficient information about counterexamples to these axioms in a fixed-size package, despite the potential largeness of the forcings involved.

Let us first introduce a convenient shorthand and recall some order-theoretic notions.

Notation 3.22. *Let $\mathbb{P}$ be a forcing notion, $p \in \mathbb{P}$ and $A \subseteq \mathbb{P}$. We say $p \leq A$ iff for all $a \in A$, $p \leq a$, and say that A is* compatible *iff there is some $q \in \mathbb{P}$ with $q \leq A$.*

Definition 3.23. *Let $\mathbb{P}$ be a forcing notion. A set $C \subseteq \mathbb{P}$ is called* **centered** *iff each finite set $A \subseteq C$ is compatible. C is* **directed** *iff for all $a, b \in C$ there is $c \in C$ with $c \leq a, b$.*

The following lemma expresses a truth about Boolean algebras. It, however, also applies to certain partial orders.

Lemma 3.24. *Let $\mathbb{P}$ be a forcing notion such that for all $p, q \in \mathbb{P}$, if p and q are compatible there is a largest lower bound, viz., some $r = p \cdot q$ with*

$r \leq p, q$ and for all $r' \leq p, q$, $r' \leq r$. Then, if $C \subseteq \mathbb{P}$ is centered, there is a filter $F \supseteq C$.

Proof. Let $\mathbb{P}$ and required an $C \subseteq \mathbb{P}$ be centered. We show that C extends to a directed set, since directed sets clearly extend to filters (by closing upwards). Recursively construct ω extensions of C: $C_0 = C$ and given C_n let $C_{n+1} = C_n \cup \{p \cdot q \mid p, q \in C\}$.

Show by induction that for each $n \in \omega$, C_n is centered. $n = 0$ is trivial. Suppose this is true for $n - 1$. Let $A \subseteq_{<\omega} C_n$. For each $a \in A$ find $p_a, q_a \in C_{n-1}$ such that $a = p_a \cdot p_a$ (if $a \in C_{n-1}$ then $p_a = q_a = a$). The set $A' = \{p_a, q_a \mid a \in A\} \subseteq C_{n-1}$ is still finite, so there is a lower bound r of A'. In particular, for each $a \in A$, $r \leq p_a, q_a$, so $r \leq p_a \cdot q_a = a$. Thus r is a lower bound for A.

Now show that $C^\omega = \bigcup_{n<\omega} C_n \supseteq C$ is directed. Let $p, q \in C^\omega$. Then $p, q \in C_n$ for some n, i.e., $p \cdot q \in C_{n+1} \subseteq C^\omega$. □

Remark 3.25. *The condition of the lemma is in particular true for Boolean algebras $\mathbb{P}$.*

Now we can state an appropriate version of the bounded fragments of PFA.

Axiom 3.26 (Bounded Fragments of PFA)**.** *Let λ be a cardinal.*

- PFA_λ *is the following axiom: Let $(\mathbb{P}, <)$ be a proper* preordered set *and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ be collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$. Then there exists a $\mathcal{D}$-generic* centered set *on $\mathbb{P}$.*

- PFA^*_λ *is the following axiom: Let $(\mathbb{P}, <)$ be a proper* Boolean algebra *and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ be collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$. Then there exists a $\mathcal{D}$-generic* filter *on $\mathbb{P}$.*

We choose this version of PFA_λ, since our method will only produce centered sets, not filters. However as we have shown, in Boolean algebras every centered sets extends to a filter. Thus our result implies PFA^*_λ. Oftentimes, one defines PFA_λ as PFA^*_λ. This is done, e.g., for BPFA=$\mathrm{PFA}_{\aleph_1}$ in [Wei08] and for $\mathrm{PFA}_{\mathfrak{c}}$ in [HJ09].

We can now define the notion of *special counterexamples*. Note that a special counterexample no longer contains an actual (potentially large) notion of forcing. For convenience, we include minimality in the definition.

Definition 3.27. *Let $\lambda > \omega$ be a cardinal. We call a triple $(D, D^*, \leq^*)$ a* **special counterexample to PFA$_\lambda$** *iff: There is a forcing notion $\mathbb{Q}$ such that:*

i. $(\mathbb{Q}, \leq)$ *is a hereditarily minimal counterexample to* PFA$_\lambda$ *(in particular, proper),*

ii. $\bigcup D \subseteq D^* \subseteq \mathbb{Q}$*,*

iii. $|D^*| \leq \lambda$, $|D| \leq \aleph_1$*,*

iv. all $A \in D$ *are predense in* $\mathbb{Q}$*,*

v. if $A \subseteq_{<\omega} D^*$ *is compatible w.r.t.* $\mathbb{Q}$*, there is* $a \in D^*$, $a \leq A$*,*

vi. $\leq^* = \leq \restriction D^*$*, and*

vii. there is no $(\mathbb{Q}, D)$*-generic centered set.*

Let Γ^λ be the class of all special counterexamples to PFA$_\lambda$*. Since the order $\leq^*$ is clear from the context in all cases, we implicitly include $\leq^*$ in D^* and w.l.o.g. consider special counterexamples as tuples (D, D^*). We shall write $\Gamma^\lambda(D, D^*, \mathbb{Q})$ if $\Gamma^\lambda(D, D^*, \leq^*)$ and $\mathbb{Q}$ witnesses that.*

The following lemma shows why this is the crucial notion for the treatment of bounded fragments of PFA. It also illustrates what we mean when we say that special counterexamples code "enough genericity".

Lemma 3.28. *Let $\lambda > \omega$, $(D, D^*, \leq^*)$ be a special counterexample to* PFA$_\lambda$ *and let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions satisfying* ii.-vi. *in the definition of special counterexamples. Let G be a filter on $\mathbb{P}$. Then $G \cap D^*$ is centered in $\mathbb{Q}$.*

Proof. We show that $g = D^* \cap G$ is centered w.r.t. $\leq^*$ in the partial order D^*. Then g is also centered w.r.t. $\mathbb{Q}$ by the definition of special counterexamples. Let $A \subseteq g$ be finite. Because G is a filter, there is $r \in G$ that is a lower bound for A. Note that it is not clear that $r \in D^*$. But, by *v.*, there is some such lower bound in D^*. □

On the other hand, we need to know that we can always find a special counterexample if we have a "normal" counterexample.

Lemma 3.29. *Let $\lambda > \omega$. If $\mathbb{Q}$ is a counterexample to* PFA_λ*, then there are D, D^* satisfying* ii.-vii. *in the definition of special counterexamples to* PFA_λ*. In particular, if $\mathbb{Q}$ is some hereditarily minimal counterexample to* PFA_λ*, then $\Gamma^\lambda(D, D^*, \mathbb{Q})$.*

Proof. Let $\mathbb{Q}$ be a counterexample to PFA_λ and let D be a set of predense sets of $\mathbb{Q}$ witnessing this. For each compatible $A \subseteq_{<\omega} \mathbb{Q}$, choose $r_A \in \mathbb{Q}$ such that $r_a \leq A$.

Let $D_0 = \bigcup D$. If D_n is defined, let $D_{n+1} = \{r_A \mid A \subseteq_{<\omega} D_n \text{ compatible}\}$. Set $D^* = \bigcup_{n\in\omega} D_n$. This process adds at most λ conditions each step, so $|D^*| \leq \lambda$. Also, if $A \subseteq_{<\omega} D^*$ is compatible w.r.t. $\mathbb{Q}$, there is some $n \in \omega$ with $A \subseteq_{<\omega} D_n$, so $r_A \in D^*$. □

Conjecture 3.30. *There is some alternative, related definition of special counterexamples that produces Lemma 3.28 for* directed *instead of centered sets.*

Argument. There might be a combinatorial notion weaker than directed but stronger than centered we can prove some version of Lemma 3.28 for, i.e., any set fulfilling such a notion would be centered and extend to a directed set. That notion then should be extendible to a filter. To that end, we could also modify the definition of special counterexamples to encode more information than mere compatibility in D^*.

Furthermore, in this chapter, we have neither used that the forcings are proper, nor that the D are predense, nor that we may formulate Lemma 3.28 for a *generic* filter. Given this wealth of untapped information, the thought of a new approach seems reasonable. □

4 Applications

4.1 PFA from a supercompact cardinal

We restate Baumgartner's classical proof of the consistency of PFA given a supercompact cardinal [Jec03, Theorem 31.21] within the framework of LHMC iterations. Most notably, we do not require a Laver function or any other kind of fast-growing function.

Since this was the very first result achieved using a LHMC iteration and we wish to give the reader the opportunity to get acquainted with the techniques used throughout this thesis, the proof of the main theorem is quite detailed. First we recall the two notions we will deal with.

Axiom 4.1 (Proper Forcing Axiom (PFA))**.** *If* $(\mathbb{P}, <)$ *is a proper forcing notion and* $\mathcal{D}$*,* $|\mathcal{D}| = \aleph_1$*, is a collection of dense subsets of* $\mathbb{P}$*, then there exists a* $\mathcal{D}$*-generic filter on* $\mathbb{P}$*.*

Definition 4.2. *A cardinal* κ *is called* λ**-supercompact** *for some cardinal* $\lambda \geq \kappa$ *iff there is a model* M *and an elementary embedding* $j : V \to M$ *such that* $\operatorname{crit}(j) = \kappa$*,* $\lambda < j(\kappa)$ *and* $M^\lambda \subseteq M$*.*

A cardinal κ *is called* **supercompact** *if it is* λ*-supercompact for all* $\lambda \geq \kappa$*.*

The following is not relevant for the argument, but it provides some insight into why our idea works at all.

Remark 4.3. *It is plausible that the LHMC iteration of* PFA *works, i.e., if* κ *is supercompact and there is a counterexample to* PFA*, there is a counterexample to* PFA *with hereditary size smaller than* κ*.*

Proof. Assume $\mathbb{P}$ is a proper forcing notion violating PFA with minimal hereditary size $\mathrm{TC}(\mathbb{P}) > \kappa$. Set $\lambda = 2^{|\mathbb{P}|}$. Take the λ-supercompactness embedding $j : V \to M$. Then, in M, $j(\mathbb{P})$ is a counterexample to PFA with minimal hereditary size $\mathrm{TC}(j(\mathbb{P})) > j(\kappa)$. As argued in Claim (i.) in Theorem 4.6 below, $\mathbb{P}$ is a proper forcing notion violating PFA with minimal hereditary size in M as well. Also by an argument there $\mathrm{TC}(\mathbb{P}) = |\mathbb{P}|$, thus $\mathrm{TC}(\mathbb{P}) < \lambda < j(\kappa) < \mathrm{TC}(j(\mathbb{P}))$. ↯ □

We will require some auxiliary properties involving models provided supercompactness and how they behave in relation to forcing.

Lemma 4.4. *Let M be a transitive model with* $\mathrm{Ord} \subseteq M$*,* $\mathbb{P} \in M$ *a* λ^+*-cc forcing notion,* G *some* $\mathbb{P}$*-generic filter on* M *and* λ *a cardinal. In* $V[G]$*, if* $V \models M^\lambda \subseteq M$ *then* $M[G]^\lambda \subseteq M[G]$*.*

Proof. Work in $V[G]$. Let $c = (c_\alpha \mid \alpha < \lambda)$ be a λ-sequence such that for all $\alpha < \lambda$, $c_\alpha \in M[G]$. For each $\alpha < \lambda$, let $\dot{c_\alpha}$ be a $\mathbb{P}$-name with $\dot{c_\alpha}^G = c_\alpha$. Let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^G = (\dot{c_\alpha} \mid \alpha < \lambda)$. Choose a $p \in G$ with $p \Vdash \forall \alpha < \check{\lambda} : \dot{a}(\alpha) \in M^{\mathbb{P}}$ in V.

Now work in V. For each $\alpha < \lambda$, there is a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$, i.e., for some $x \in M$, $q \Vdash \dot{a}(\alpha) = \check{x}$. Define $\sigma = \{((\check{\alpha, x}), q) \mid \alpha < \lambda, q \in A_\alpha, q \Vdash \dot{a}(\alpha) = \check{x}\}$. Then $p \Vdash \sigma = \dot{a}$. Notice that $|\sigma| \leq \lambda$, since for each α, $|A_\alpha| \leq \lambda$. Thus $\sigma \in M$.

Work in $V[G]$ again. $(\dot{c_\alpha} \mid \alpha < \lambda) = \dot{a}^G = \sigma^G \in M[G]$. We can compute $c = (c_\alpha \mid \alpha < \lambda) = (\dot{c_\alpha}^G \mid \alpha < \lambda)$ from $(\dot{c_\alpha} \mid \alpha < \lambda)$ and G. Hence by Replacement, $c \in M[G]$. □

Lemma 4.5. *Let* λ *be a cardinal and* $M^\lambda \subseteq M$ *for some model* M *with* $\mathrm{Ord} \subseteq M$*. Then* $H^M_{\lambda^+} \supseteq H_{\lambda^+}$*.*

Proof. Let $x \in H_{\lambda^+}$ and set $a := |\mathrm{TC}(\{x\})| \leq \lambda$. Find a bijection $f : |\mathrm{TC}(\{x\})| \to \mathrm{TC}(\{x\})$ with $f(\emptyset) = x$. Now define a relation R on a^2 by $\alpha R \beta \leftrightarrow f(\alpha) \in f(\beta)$.

Then, (a, R) has some transitive collapse in $a^2 \subseteq \lambda$. By assumption $M^\lambda \subseteq M$, i.e., $a^2, R \in M$. We can reconstruct x from a^2 and R. □

Now we are able to state the central result of this section. The proof involves many techniques important for other applications.

Theorem 4.6. *If* κ *is* λ*-supercompact, then the LHMC iteration of* PFA*,* $\mathbb{P}_\kappa$*, forces that* PFA *holds for all proper forcings* $\mathbb{P}$ *with* $2^{|\mathbb{P}|} \leq \lambda$*.*

Proof. We closely follow Baumgartner's argument. Let $j \colon V \to M$ be a λ-supercompactness embedding, i.e., $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $M^\lambda \subseteq M$.

Assume the theorem is false. Suppose that G is $\mathbb{P}_\kappa$-generic over V. We work in $V[G]$. Let $\mathbb{P}$ be a proper forcing violating PFA with $2^{|\mathbb{P}|} \leq \lambda$ of minimal hereditary size. Let $\mathcal{D} = \{D_\alpha \mid \alpha < \aleph_1\}$ witness this. We show that $\mathbb{P} \in M[G]$ by Lemma 4.5, since $M[G]^\lambda \subseteq M[G]$ by Lemma 4.4:

$\mathbb{P}_\kappa$ satisfies the λ^+-cc by Corollary 3.11, so to apply Lemma 4.4 it remains to show that $\mathbb{P}_\kappa \in M$; it suffices to show that $\mathbb{P}_\kappa \subseteq M$. So, let $p \in \mathbb{P}_\kappa$. Since

$\mathbb{P}_\kappa$ is a countable support iteration, there is some $\gamma < \kappa$ such that $p(\alpha) = \mathbb{1}$ for all $\alpha > \gamma$. Since $j(\gamma) = \gamma$, $j(p)(\alpha) = \mathbb{1}$ for all $\alpha > \gamma$. Furthermore, $p(\alpha) = (p \restriction \gamma)(\alpha)$ for all $\alpha \leq \gamma$, hence

$$j(p)(\alpha) = j(p \restriction \gamma)(\alpha) = (p \restriction \gamma)(\alpha) = p(\alpha)$$

for all $\alpha \leq \gamma$. Thus $j(p) = p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1}$, i.e., $j(p) \restriction \kappa = p \in M$.

Claim (i)**.** *In $M[G]$, $\mathbb{P}$ violates* PFA*, is of minimal hereditary size with that property and $\mathbb{P} \in H_{j(\kappa)}$.*[3]

Proof. $|\mathrm{TC}(\mathbb{P})| = |\mathbb{P}|$: Suppose not, take a bijection $f : \mathbb{P} \to \alpha = |\mathbb{P}|$ and define a relation $<_\alpha$ on α by $\beta <_\alpha \gamma$ iff $f^{-1}(\beta) <_{\mathbb{P}} f^{-1}(\gamma)$. $(\alpha, <_\alpha)$ is a forcing notion equivalent to $\mathbb{P}$ but of smaller hereditary size $\mathrm{TC}(\alpha) = \alpha$. ↯

We now show that $\mathbb{P}$ is proper in $M[G]$. Let $\mu = (|\mathbb{P}|)^+$. Since we now know $|\mathrm{TC}(\mathbb{P})| = |\mathbb{P}| < \mu$, $\mathbb{P} \in H_\mu$. Choose a club $C \subseteq [H_\mu]^\omega$ witnessing that $\mathbb{P}$ is proper in $V[G]$. Note (using Lemma 2.4):

$$|C| \leq |H_\mu| \leq 2^{<\mu} \leq 2^{|\mathbb{P}|} \leq \lambda.$$

Therefore by Lemma 4.5, $C \in M[G]$ and hence C witnesses that $\mathbb{P}$ is proper in $M[G]$.

Also, $V[G]$ and $M[G]$ have the same $\aleph_1$ (namely, $\aleph_1^V = \aleph_1^M$), since $\mathbb{P}_\kappa$ is proper (as a countable support iteration of proper forcing notions). Hence, $|\mathcal{D}|^{M[G]} = \aleph_1^{M[G]}$. For all $\alpha < \omega_1$, $D_\alpha \subseteq \mathbb{P} \in M[G]$, $|D_\alpha| \leq |\mathbb{P}| \leq \lambda$, i.e., $D_\alpha \in M[G]$. Thus, since $\aleph_1 < \lambda$, $\mathcal{D} \in M[G]$.

Furthermore, $|\mathrm{TC}(\mathbb{P})| < \lambda < j(\kappa)$, so $\mathbb{P} \in H_{j(\kappa)}$. Finally, if there were a hereditary smaller counterexample in $M[G]$, it would be in $V[G]$ and be a counterexample to PFA there, because $M[G]$ is sufficiently closed to contain filters witnessing the contrary and clubs witnessing properness. Hence this would contradict the hereditarily minimality of $\mathbb{P}$. □

In M, the forcing $j(\mathbb{P}_\kappa)$ is, by elementarity, a countable support iteration of length $j(\kappa) > \lambda$ and $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, since $\mathrm{crit}(j) = \kappa$ (i.e. $j \restriction H_\kappa = \mathrm{id}$ while $\mathbb{P}_\alpha \in H_\kappa$ for all $\alpha < \kappa$). By the Factor Lemma [Jec03, Lemma 21.8], $j(\mathbb{P}_\kappa)$ is forcing equivalent to an iteration $(\mathbb{P}_\kappa * \dot{\mathbb{P}}) * \dot{\mathbb{P}}_{\kappa, j(\kappa)}$ where

[3] Many models used in further applications will satisfy some version of that claim and we refer to (parts of) the proof as "usual arguments". In particular we will always show that these models verify enough properness and are sufficiently closed.

$\dot{\mathbb{P}}^G$ is the lottery sum of all counterexamples to PFA in $M[G]$ of minimal hereditary size smaller $j(\kappa)$.

Let H be $\mathbb{P}$-generic over $V[G]$. Note that there is a $\dot{\mathbb{P}}^G$-generic $\tilde{H}$ over $M[G]$ with $M[G * H] = M[G * \tilde{H}]$. Let I be $\dot{\mathbb{P}}^{G*\tilde{H}}_{\kappa,j(\kappa)}$-generic over $V[G * \tilde{H}]$.

We now work in $V[(G * \tilde{H}) * I]$. Consider:

$$j^*\colon V[G] \to M[(G * \tilde{H}) * I],$$

$$j^*(\sigma^G) = j(\sigma)^{(G*\tilde{H})*I}.$$

Claim (ii)**.** *j^* is well-defined and elementary and extends j.*

Proof. Well-defined: Let σ, τ be $\mathbb{P}_\kappa$-names with $\sigma^G = \tau^G$. Then there is $p \in G$ such that $p \Vdash \sigma = \tau$, i.e., $j(p) \Vdash j(\sigma) = j(\tau)$. $j(p)$ is an element of $(G * \tilde{H}) * I$: $p = (p_\alpha \mid \alpha < \kappa)$ with countable support, so there is some $\beta < \kappa$ with $p_\gamma = \mathbb{1}$ for all $\gamma \geq \beta$. $V[G] \models \forall\gamma < \beta : p(\gamma) = (p \restriction \beta)(\gamma)$, so

$$\forall\gamma < j(\beta) : j(p)(\gamma) = (j(p \restriction \beta))(\gamma).$$

Since $j \restriction H_\kappa = \mathrm{id}$, $\mathbb{P}_\gamma \in H_\kappa$ and $j(\beta) = \beta$, $j(p)(\gamma) = p(\gamma)$ below β and $\mathbb{1}$ otherwise. Therefore $j(p) = p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1} \in (G * \tilde{H}) * I$.

Elementarity: Let $\varphi = \varphi(x)$ be a formula, σ a $\mathbb{P}_\kappa$-name and suppose $V[G] \models \varphi(\sigma^G)$. Then there is some $p \in G$ with $p \Vdash \varphi(\sigma)$, i.e., $j(p) \Vdash \varphi(j(\sigma))$. As above $j(p) \in (G * \tilde{H}) * I$.

Extension: Trivial (use canonical names). □

Suppose that $\mathcal{D}$ is a family of size $\aleph_1$ of dense subsets of $\mathbb{P}$ in $V[G]$. As in (i), $\mathcal{D}$ is a family of size $\aleph_1$ of dense subsets of $\mathbb{P}$ in $M[G]$. We show that there is a $(j^*(\mathbb{P}), j^*(\mathcal{D}))$-generic filter in $M[(G*\tilde{H})*I]$. Notice that $j^* \restriction \mathbb{P} \in M[G]$, since $|\mathbb{P}| < \lambda$. $H \subseteq \mathbb{P}$ and therefore by Replacement $j^*[H] \in M[(G * \tilde{H}) * I]$.

Recall that $V[G]$ and $M[(G * \tilde{H}) * I]$ agree on $\aleph_1$, i.e., $j^*(\omega_1) = \omega_1$. Now show $j^*(\mathcal{D}) = \{j^*(D) \mid D \in \mathcal{D}\}$: There is a enumeration $f : \omega_1 \to \mathcal{D}$ and since ω_1 is not changed, $j^*(f) : \omega_1 \to j^*(\mathcal{D})$ enumerates $j^*(\mathcal{D})$. Let $A \in j^*(\mathcal{D})$. Then there is $\alpha < \omega_1$ with $A = j^*(f)(\alpha) = j^*(f(\alpha)) = j^*(D)$ for some $D \in \mathcal{D}$.

H is $\mathbb{P}$-generic in $V[G]$, in particular it intersects every $D \in \mathcal{D}$. Thus for every $D \in \mathcal{D}$ there is some $x_D \in H$ such that $V[G] \models x_D \in D$, so by elementarity, $M[(G * H) * I] \models j^*(x_D) \in j^*(D)$.

Therefore the filter on $j^*(\mathbb{P})$ generated by $j^*[H]$ in $M[(G * \tilde{H}) * I]$ intersects every $D \in j^*(\mathcal{D})$, i.e., it is $(j^*(\mathbb{P}), j^*(\mathcal{D}))$-generic. Hence, by elementarity, there is a $(\mathbb{P}, \mathcal{D})$-generic filter in $V[G]$. $\square$

The classical result follows immediately.

Corollary 4.7. *If κ is a supercompact cardinal, then $\mathbb{P}_\kappa$ forces* PFA*, hence* PFA *is consistent relative to the existence of a supercompact cardinal.*

4.2 SPFA from a supercompact cardinal

The consistency of SPFA given a supercompact cardinal is a result of Foreman, Magidor and Shelah [FMS88]. The typical proof is nearly the same as Baumgartner's classical argument and uses a Laver function [Jec03, Theorem 37.9]. Since the main effort in this proof is the investigation of RCS iterations, we merely illustrate how the LHMC iteration adapts similar to the classical Laver-style iteration of Baumgartner.

Axiom 4.8 (Semiproper Forcing Axiom (SPFA))**.** *If $(\mathbb{P}, <)$ is a semiproper forcing notion and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$, is a collection of dense subsets of $\mathbb{P}$, then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.*

Theorem 4.9. *If κ is λ-supercompact, then $\mathbb{P}_\kappa$ forces that* SPFA *holds for all semiproper forcings $\mathbb{P}$ with $2^{|\mathbb{P}|} \leq \lambda$.*

The proof is exactly the same as Theorem 4.6. We use the characterization of semiproperness in Lemma 2.33. Whenever we use the countable supports of $\mathbb{P}_\kappa$, we actually only require that $\mathbb{P}_\kappa$ is a direct limit. This is given by Corollary 3.12.

Other required properties of properness / countable support also hold, in particular ω_1 is preserved (Corollary 2.32).

4.3 BPFA from a reflecting cardinal

BPFA was first described and shown to be consistent relative to a reflecting cardinal by Goldstern and Shelah [GS95]. Their original argument involved some intricate combinatorial equivalences which we do not claim to fully understand. Instead, we prove the consistency result within the framework of LHMC iterations.

An important method are the *special counterexamples* defined above. They allow us to code "enough genericity" of a possible counterexample to BPFA to produce that genericity in our iteration. Another proof can be found in [Wei08, Theorem 4.6]. The techniques there share a noticable degree of similarity with our special counterexamples.

Axiom 4.10 (Bounded Proper Forcing Axiom (BPFA))**.** BPFA *is* $\mathrm{PFA}_{\aleph_1}$*, viz.: If* $(\mathbb{P}, <)$ *is a proper forcing notion and* $\mathcal{D}$*,* $|\mathcal{D}| = \aleph_1$*, is a collection of predense sets of size at most* ω_1 *in* $\mathbb{P}$*, then there exists a* $\mathcal{D}$*-generic* centered set *on* $\mathbb{P}$*.*

Definition 4.11. *A cardinal* κ *is* **reflecting** *iff it is regular and for any formula* φ *and any* $a \in H_\kappa$*, if there is a cardinal* $\delta > \kappa$ *with* $H_\delta \models \varphi(a)$*, then there is some cardinal* $\gamma < \kappa$ *with* $a \in H_\gamma$ *and* $H_\gamma \models \varphi(a)$*.*

Remark 4.12. *This definition is also sometimes known as "*Σ_2*-reflecting" or "*Σ_2*-correct". Reflecting cardinals are below Mahlo cardinals in consistency strength.*

Proof. Show: If κ is inaccessible, then $\{\alpha < \kappa \mid V_\alpha \prec V_\kappa\}$ is club in κ. Show first unboundedness (and non-emptiness): Let $\alpha < \kappa$ be arbitrary and define a sequence by induction: $\alpha_0 = \alpha$. Suppose α_n is known. Let $\alpha_{n+1} \geq \alpha_n$ such that for all formulae φ and all $\bar{y} \in V_{\alpha_n}$, if $V_\kappa \models \exists x \varphi(x, \bar{y})$, then there is $\tilde{x} \in V_{\alpha_{n+1}}$ such that $V_\kappa \models \varphi(\tilde{x}, \bar{y})$. Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$, hence there are less than κ many such $\tilde{x}$, i.e., $\alpha_{n+1} < \kappa$.

Define $\hat{\alpha} = \sup_{n<\omega} \alpha_n \geq \alpha$. Then obviously $V_\alpha \prec V_\kappa$ by the Tarski-Vaught criterion. Closure is trivial, since if $V_{\gamma_n} \prec V_\kappa$, $n < \omega$, then again by Tarski-Vaught, $\bigcup_{n<\omega} V_{\gamma_n} \prec V_\kappa$.

Now let μ be Mahlo. In V_μ, there is a club $C \subseteq \{V_\alpha \mid \alpha < \mu\}$ of elementary submodels of V_δ. Let $\kappa \in C$ be inaccessible. Then V_μ models that κ is reflecting: If $a \in H_\kappa$, $\delta \in V_\mu$ and $V_\mu \models$ "$H_\delta \models \varphi(a)$", then $V_\mu \models \exists \delta :$ "$V_\delta \models \varphi(a)$". By elementarity, so does κ. □

We observe that reflecting cardinals are indeed large.

Lemma 4.13. *If* κ *is a reflecting cardinal,* $\kappa > \aleph_1$ *and* κ *is inaccessible.*

Proof. Suppose $\aleph_1$ is reflecting. Assume $\kappa \leq \aleph_1$. Then, all sets in H_γ for any cardinal $\gamma < \kappa$ are finite (γ is at most ω). So we can't reflect the Axiom of Infinity.

Assume κ is a successor. Then $\kappa = \delta^+$, i.e., $\delta \in H_\kappa$. Then there is some $\gamma < \kappa$, i.e., $\gamma \leq \delta$, with $\delta \in H_\gamma$ $\lightning$.

Assume there is some $\delta < \kappa$ with $2^\delta \geq \kappa$. Then $\delta \in H_\kappa$ and we may reflect "2^δ exists". So, there is some $\gamma < \kappa$ such that $H_\gamma \models$ "2^δ exists". But $2^\delta \notin H_\gamma$ $\lightning$. $\square$

Remark 4.14. *By adding any cardinal $\alpha < \kappa$ as a parameter to φ, we can make the γ provided by the reflecting property as large as we require.*

The next result shows that reflecting cardinals are indestructible by small forcing. We make pivotal use of our preliminary work on hereditary sets.

Lemma 4.15. *Let $\mathbb{P} \in H_\kappa$. If κ is reflecting, then $\mathbb{1}_\mathbb{P} \Vdash$ "κ is reflecting".*

Proof. Let $\mathbb{P} \in H_\kappa$. Let φ be a formula with

$$\mathbb{1}_\mathbb{P} \Vdash \text{"}\dot{a} \in H_\kappa \wedge \exists \delta > \kappa : H_\delta \models \varphi(\dot{a})\text{"}.$$

W.l.o.g. (Lemma 2.9) we may assume that $\dot{a} \in H_\kappa$. By Theorem 2.10: $H_\delta \models \mathbb{1}_\mathbb{P} \Vdash \varphi(\dot{a})$. Since κ is reflecting, there is $\gamma < \kappa$ such that $\mathbb{P}, \dot{a} \in H_\gamma$ and $H_\gamma \models \mathbb{1}_\mathbb{P} \Vdash \varphi(\dot{a})$. So by Theorem 2.10, $\mathbb{1}_\mathbb{P} \Vdash H_\gamma \models \varphi(\dot{a})$.

Since $\mathbb{P} \in H_\gamma$, i.e., $\mathbb{P}$ satisfies the γ-cc, γ remains a cardinal. Thus κ is reflecting in any forcing extension. $\square$

Furthermore, reflecting cardinals provide small witnesses to special counterexamples to BPFA. Combined with the previous result this will be the crucial step in our main argument.

Lemma 4.16. *Let κ be reflecting. If there is a special counterexample D, D^* to* BPFA*, $D, D^* \in H_\kappa$, then there is a forcing $\mathbb{Q}$ witnessing this in H_κ.*

Proof. Suppose there are $D, D^* \in H_\kappa$ with $\Gamma^{\aleph_1}(D, D^*)$. There is a cardinal δ such that

$$H_\delta \models \text{"}\Gamma^{\aleph_1}(D, D^*),\ \exists \lambda : \exists \mathbb{Q} \in H_\lambda : \Gamma^{\aleph_1}(D, D^*, \mathbb{Q}), 2^\lambda \text{ exists"}.$$

Since κ is reflecting, there is $\gamma < \kappa$ such that

$$H_\gamma \models \text{"}\Gamma^{\aleph_1}(D, D^*),\ \exists \lambda : \exists \mathbb{Q} \in H_\lambda : \Gamma^{\aleph_1}(D, D^*, \mathbb{Q}), 2^\lambda \text{ exists"}.$$

Choose such $\mathbb{Q}$ and λ, i.e., $H_\gamma \models$ "$\mathbb{Q} \in H_\lambda$, 2^λ exists, $\Gamma^{\aleph_1}(D, D^*, \mathbb{Q})$". By Lemma 2.22, $\mathbb{Q}$ is really proper. All other properties of "special counterexample to BPFA" are obviously absolute. □

Now we can show the main result in this section. Note that the proof makes use of the notion of *special counterexamples* we introduced in the preliminaries.

Theorem 4.17. *If a reflecting cardinal κ exists, the LHMC iteration of* BPFA, $\mathbb{P}_\kappa$, *forces* BPFA*.*

Proof. Suppose not. Let p be some condition that forces $\Gamma^{\aleph_1} \neq \emptyset$. Let G be $\mathbb{P}$-generic over V, $p \in G$ and live in $V[G]$. Take witnesses (viz., a special counterexample to BPFA) D, D^* for $\Gamma^{\aleph_1}$. Note that $\omega_1^V = \omega_1^{V[G]}$ since $\mathbb{P}_\kappa$ is proper. Also $\mathbb{P}_\kappa$ does not collapse κ.

Since D, D^* are of size at most ω_1 and we can consider the forcing witnessing $\Gamma^{\aleph_1}(D, D^*)$ as a subset of an ordinal, we may assume $D^* \subseteq \omega_1$, $\leq^* \subseteq \omega_1^2$ and for each $A \in D$, $A \subseteq \omega_1$. Enumerate $D = (A_\alpha)_{\alpha<\omega_1}$, and let $\tilde{D} = \{(x, \alpha) \mid x \in A_\alpha\} \subseteq \omega_1^2$. Clearly, we can recompute D from $\tilde{D}$, hence w.l.o.g. assume $D \subseteq \omega_1^2$. Therefore, since κ is regular and not collapsed, we may apply Lemma 2.26 and find some $\alpha < \kappa$ with $D, D^* \in V[G_\alpha]$, $G_\alpha = \{q \upharpoonright \alpha \mid q \in G\}$.

G_α is $\mathbb{P}_\alpha$-generic and $\mathbb{P}_\alpha \in H_\kappa$ by the construction of $\mathbb{P}_\kappa$, hence by Lemma 4.15, κ is reflecting in $V[G_\alpha]$. Now work in $V[G_\alpha]$.

Because $\mathbb{P}_\kappa$ is a countable support iteration, there is some $q \in H_\kappa^V \subseteq H_\kappa$ such that $p = q^\frown \mathbb{1}^\kappa$. The statement $\exists \lambda : q^\frown \mathbb{1}^\lambda \Vdash_{\mathbb{P}_\lambda} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$ holds (take $\lambda = \kappa$) and its parameters are in H_κ. So, since κ is reflecting, there are $\gamma < \delta < \kappa$ with $H_\delta \models q^\frown \mathbb{1}^\gamma \Vdash_{\mathbb{P}_\gamma} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$, and since this is Σ_2, by Lemma 2.7, $q^\frown \mathbb{1}^\gamma \Vdash_{\mathbb{P}_\gamma} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$ is true.

$\mathbb{P}_\gamma$ has hereditary size smaller κ, thus $q^\frown \mathbb{1}^\gamma$ also forces that κ is reflecting, thus it forces that there is a witness $\mathbb{Q}$ to $\Gamma^{\aleph_1}(D, D^*)$ with hereditary size smaller κ by Lemma 4.16. W.l.o.g. assume $\mathbb{Q}$ has minimal hereditary size; then there is some $r \leq q^\frown \mathbb{1}^\kappa = p$ choosing that $\mathbb{Q}$ from the lottery sum in the γ-th step.

Hence forcing with r adjoins a $\mathbb{Q}$-generic filter h, so $h \in V[G]$ for each generic G with $r \in G$. h intersects each $A \in D$ and is a filter on D^*. Thus by the construction of D^*, $h \cap D^*$ extends to a centered set on *any* witness to $\Gamma^{\aleph_1}(D, D^*)$. Hence r forces $\neg\Gamma^{\aleph_1}(D, D^*)$ contradicting the assumption that $p \Vdash \Gamma^{\aleph_1}(D, D^*)$. □

4.4 BSPFA from a reflecting cardinal

While BSPFA has received much less attention than SPFA, we also quickly illustrate that our proof for BPFA generalizes to semiproper forcing and RCS iterations as it does with PFA/SPFA. A notable fact about BSPFA, which may also be why it has not received much attention, is that it is much weaker than BMM, see [Sch04].

Axiom 4.18 (Bounded Semiproper Forcing Axiom (BSPFA))**.** *If* $(\mathbb{P}, <)$ *is a semiproper notion of forcing and* $\mathcal{D}$*,* $|\mathcal{D}| = \aleph_1$*, is a collection of predense sets of size at most* ω_1 *in* $\mathbb{P}$*, then there exists a* $\mathcal{D}$*-generic* centered set *on* $\mathbb{P}$*.*

Lemma 4.19. *Let* $\mathbb{P}$ *be a forcing notion,* $2^{|\mathbb{P}|} < \delta$ *and* $\mathbb{P} \in H_\delta$*. Then:* $\mathbb{P}$ *is semiproper iff* $H_\delta \models$ "$\mathbb{P}$ is semiproper"*.*

Proof. Just as Lemma 2.22, using Lemma 2.33 instead of Corollary 2.20. □

Theorem 4.20. *If a reflecting cardinal* κ *exists, the LHMC iteration of* BSPFA*,* $\mathbb{P}_\kappa$*, forces* BSPFA*.*

The proof works exactly as in Theorem 4.17. The obvious analog definition of special counterexamples to BSPFA works the same, and instead of Lemma 2.26 we can use Lemma 2.47.

4.5 $MA_{\aleph_1}$ from an inaccessible cardinal

To apply LHMC iterations to axioms outside the scope of PFA, we attempted a novel consistency proof of Martin's Axiom. However, since the niceness/smallness properties of LHMC iterations are dependent on a large cardinal, we were unable to completely reproduce the classical result that MA is consistent relative to ZFC. We tried to discuss some reasons for our failure to do so.

Nevertheless, this investigation proved worthwhile after all, since it is well-known that $MA_{\mathfrak{c}}$ fails. Therefore this section provides some additional insight in what LHMC iterations do with the continuum.

Axiom 4.21 (Martin's Axiom (MA))**.** $\mathrm{MA}_{\aleph_1}$ *is the following axiom: If* $(\mathbb{P}, <)$ *is a ccc forcing notion and* $\mathcal{D}$*,* $|\mathcal{D}| \leq \aleph_1$*, is a collection of dense subsets of* $\mathbb{P}$*, then there exists a* $\mathcal{D}$*-generic filter on* $\mathbb{P}$*.*

The following lemma is the crucial observation. It mimics the techniques applied to find special counterexamples, but the ccc allows it to be much stronger.

Lemma 4.22. *If there is a counterexample to* $\mathrm{MA}_{\aleph_1}$*, there is a counterexample to* $\mathrm{MA}_{\aleph_1}$ *of size* ω_1*.*

Proof. Suppose $\mathbb{P}$ is a counterexample to $\mathrm{MA}_{\aleph_1}$. Let $\mathcal{D}$, $|\mathcal{D}| \leq \aleph_1$ be a set of dense subsets of $\mathbb{P}$ witnessing this. For each $D \in \mathcal{D}$ let $A_D \subseteq D$ be a maximal antichain in $\mathbb{P}$. Notice that $|A_D| \leq \omega$.

Construct an ω-sequence of forcings $(\mathbb{Q}_n)_{n\in\omega}$: $\mathbb{Q}_0 = \bigcup_{D\in\mathcal{D}} A_D \cup \{\mathbb{1}\}$. Suppose $\mathbb{Q}_n$ is known. Let ("compatible" in particular includes "equal")

$$Q = \{(p, q) \in (\mathbb{Q}_n)^2 \mid p \text{ and } q \text{ are compatible in } \mathbb{P}\}.$$

For each $(p, q) \in Q$ and each $D \in \mathcal{D}$ choose some $r^D_{p,q} \in D$ such that $p, q \geq r^D_{p,q}$ (this is possible, since the $D \in \mathcal{D}$ are dense). Then we can define $\mathbb{Q}_{n+1} = \{r^D_{p,q} \mid (p, q) \in Q, D \in \mathcal{D}\} \cup \mathbb{Q}_n$.

This process adds ω_1 new elements in each step, so $\mathbb{Q} = \bigcup_{n\in\omega} \mathbb{Q}_n$ has size ω_1. Also $\mathbb{Q}$ is ccc, since by construction all its antichains are antichains of $\mathbb{P}$. Let $\tilde{\mathcal{D}} = \{D \cap \mathbb{Q} \mid D \in \mathcal{D}\}$ and suppose there is a $(\mathbb{Q}, \tilde{\mathcal{D}})$-generic filter. By construction it can be extended (closing upwards) to a $(\mathbb{P}, \mathcal{D})$-generic filter. The $\tilde{D} \in \tilde{\mathcal{D}}$ are dense in $\mathbb{Q}$ by construction and hence $\mathbb{Q}$ is a counterexample to $\mathrm{MA}_{\aleph_1}$. □

Now we can show the consistency of MA from an inaccessible. We can again draw on the groundwork done in the preliminaries.

Theorem 4.23. *Let* κ *be an inaccessible cardinal. Then the LHMC iteration of* $\mathrm{MA}_{\aleph_1}$*,* $\mathbb{P}_\kappa$*, of length* κ *forces* $\mathrm{MA}_{\aleph_1}$*.*

Proof. Assume not. Let $p \in \mathbb{P}_\kappa$ such that p forces that there is a counterexample to $\mathrm{MA}_{\aleph_1}$. Let G, $p \in G$, be $\mathbb{P}_\kappa$-generic and let $\mathbb{P}, \mathcal{D}$ be a hereditarily minimal counterexample to $\mathrm{MA}_{\aleph_1}$ of size ω_1 in $V[G]$. W.l.o.g. we can assume $\mathbb{P} \subseteq \omega_1$, $\leq_\mathbb{P} \subseteq \omega_1^2$ and $\mathcal{D} \subseteq \omega_1^2$ by encoding $\mathcal{D} = (A_\alpha)_{\alpha<\omega_1} \simeq \{(x, \alpha) \mid x \in A_\alpha\}$

if necessary. Note that $\mathbb{P}_\kappa$ is proper, i.e., $p \Vdash \omega_1 = \omega_1^V$. Now recall Lemma 2.26.

Since $\mathbb{P}_\kappa$ satisfies the κ-cc, $\omega_1^V < \mathrm{cf}^{V[G]}\,\kappa$, so there is some $\gamma < \kappa$ with $\mathbb{P}, \mathcal{D} \in V[G_\gamma]$. W.l.o.g. assume supp$(p) \subseteq \gamma$. Clearly, $\mathbb{P}$ is ccc in $V[G_\gamma]$ and if there were a $\mathbb{P}, \mathcal{D}$-generic filter in $V[G_\gamma]$ it would be in $V[G]$. So $\mathbb{P}, \mathcal{D}$ is a counterexample to MA$_{\aleph_1}$ in $V[G_\gamma]$.

ω_1 is preserved along the iteration, so $\mathbb{P}$ retains it's cardinality ω_1 in $V[G_\gamma]$. Thus – there are no countable counterexamples to MA$_{\aleph_1}$ – $\mathbb{P}$ is a minimal counterexample, and hence there is some $q \leq p$ choosing $\mathbb{P}$ from the lottery sum in the γ-th step. By the Factor Lemma, q forces that there is a generic filter on $\mathbb{P}$. □

Since it is well known that MA$_{\aleph_1}$ is consistent with ZFC alone, one might ask if it is really necessary to use an inaccessible cardinal.

Conjecture 4.24 (Schlicht)**.** *An iteration of length $\aleph_2$ suffices.*

This is still unresolved and we do not fully understand the circumstances where it might hold or fail. Nevertheless, if this should turn out to be true, it would require different techniques than we have used so far.

Remark 4.25. *We cannot drop the large cardinal assumption with the current approach.*

Argument. To apply Lemma 2.26, we require that the iteration of length κ does not collapse κ and its cofinality. We use the κ-cc for this. Corollary 3.11, giving the chain condition, requires κ to be regular.

At each step $\alpha < \kappa$ in the iteration, $\mathbb{Q}_\alpha$ is a lottery sum of forcings of size ω_1, i.e., there are potentially 2^{ω_1} such forcings. So $\mathbb{Q}_\alpha$ can have antichains of size $(2^{\omega_1})^{V[G_\alpha]}$. In particular, in the best case (GCH) $\mathbb{P}_{\alpha+1}$ satisfies the $\aleph_3$-cc.

Since ccc forcings can add Cohen subsets, the size of 2^{ω_1} in $V[G_\alpha]$ is only bounded by the chain condition of $\mathbb{P}_\alpha$ as in Lemma 2.24. So, if $\mathbb{P}_\alpha$ satisfies the $\aleph_3$-cc, the best result achievable by this means (and assuming GCH) is $(2^{\omega_1})^{V[G_\alpha]} \leq 2^{\omega_2} < \aleph_4$.

Thus, chain-condition-wise, we must go larger in each step, i.e., we would require a limit cardinal. But since we also require a regular cardinal, we indeed require an inaccessible.

Of course, more sophisticated results to keep the size of each step small or to preserve chain conditions in iterations of *singular* length would fix this. Fixed points of $\aleph$ with large enough cofinality seem to be viable candidates for such approaches, but we were unable to make it work. Another possible approach would be to show that the iteration of length ω_2 does not collapse ω_2 without verifying the ω_2-cc. Maybe using additional assumptions about our ground model. $\square$

We shall apply our results about MA to other iterations. Again there are some auxiliary results.

Lemma 4.26. *Any counterexample to* $MA_{\aleph_1}$ *is a ccc counterexample to* (B)PFA*.*

Proof. PFA is clear. Let $\mathbb{P}, \mathcal{D}$ be a counterexample to $MA_{\aleph_1}$, in particular let $\mathbb{P}$ be ccc. For each $D \in \mathcal{D}$ let $E \subseteq D$ be a maximal antichain and let $\mathcal{E}$ be the collection of all these E. Then $\mathcal{E}$ is a collection of predense sets of size less than ω_1 in $\mathbb{P}$ and there is no $(\mathbb{P}, \mathcal{E})$-generic filter in $\mathbb{P}$, i.e., $\mathbb{P}$ is a counterexample to BPFA. $\square$

Corollary 4.27. *If there is a ccc counterexample to* (B)PFA*, there is a ccc counterexample to* (B)PFA *of size* $\aleph_1$*.*

Proof. Any ccc counterexample to (B)PFA is in particular a counterexample to $MA_{\aleph_1}$. By Lemma 4.22, there is then a counterexample to $MA_{\aleph_1}$ of size ω_1. And by Lemma 4.26, that counterexample to $MA_{\aleph_1}$ is a ccc counterexample to (B)PFA. $\square$

Now we can infer that the LHMC iterations of (B)PFA will, if they are nice (i.e. of large cardinal length), negate CH.

Corollary 4.28. *If* κ *is inaccessible, the LHMC iteration of* (B)PFA*,* $\mathbb{Q}_\kappa$*, forces* $MA_{\aleph_1}$*. In particular,* $\mathbb{Q}_\kappa$ *forces* $\mathfrak{c} > \aleph_1$*.*

Proof. The proof of Theorem 4.23 with Corollary 4.27 shows that the LHMC iterations of (B)PFA force that there are no ccc counterexamples to (B)PFA. By Lemma 4.26 this implies $MA_{\aleph_1}$. Since $MA_{\mathfrak{c}}$ is false, this concludes the proof. $\square$

4.6 PFA$^-$ from an inaccessible cardinal

This section is a strengthening of the previous one, and the main theorem is a strengthening of Theorem 4.23 using the same techniques. Just as Martin's Axiom, PFA$^-$ has been proven to be consistent with ZFC [Jec03, Exc. 31.10], so our approach falls one inaccessible cardinal short of the classical result.

Axiom 4.29 (PFA$^-$). *If* $(\mathbb{P}, <)$, $|\mathbb{P}| \leq \aleph_1$ *is a proper forcing notion and* $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$, *is a collection of dense subsets of* $\mathbb{P}$, *then there exists a* $\mathcal{D}$*-generic filter on* $\mathbb{P}$.

Theorem 4.30. *Let* κ *be inaccessible. Then the LHMC iteration of* (B)PFA $\mathbb{P}_\kappa$ *with length* κ *forces that there are no counterexamples to* (B)PFA *of size* ω_1.

Proof. We follow the proof of Theorem 4.23. So assume the theorem is false. Let $p \in \mathbb{P}_\kappa$ such that p forces that there is a counterexample to (B)PFA of size ω_1. Let G, $p \in G$ be $\mathbb{P}_\kappa$-generic and let $\mathbb{P}, \mathcal{D}$ be a hereditarily minimal counterexample to (B)PFA of size ω_1 in $V[G]$. Again we may assume $\mathbb{P} \subseteq \omega_1$ and $\mathcal{D} \subseteq \omega_1^2$. Recall that $\mathbb{P}_\kappa$ is proper, i.e., $p \Vdash \omega_1 = \omega_1^V$. Now we apply Lemma 2.26.

Since $\mathbb{P}_\kappa$ satisfies the κ-cc, $\omega_1^V < \mathrm{cf}^{V[G]}\,\kappa$, there is some $\gamma < \kappa$ with $\mathbb{P}, \mathcal{D} \in V[G_\gamma]$. W.l.o.g. assume $\mathrm{supp}(p) \subseteq \gamma$. Now show that $\mathbb{P}$ is proper in $V[G_\gamma]$. If not, there is an uncountable cardinal λ and stationary $S \subseteq [\lambda]^\omega$ in $V[G_\gamma]$ such that $\mathbb{P}$ does not preserve S's stationarity. Because ω_1 is not collapsed, λ is uncountable in $V[G]$. $\mathbb{P}_\kappa$ factors into $\mathbb{P}_\gamma * \mathbb{P}_{\gamma,\kappa}$ and $\mathbb{P}_{\gamma,\kappa}$ is proper, i.e., S is stationary in $V[G]$.

Now let H be $\mathbb{P}$-generic over $V[G_\gamma]$ such that in $V[G_\gamma][H]$, S is not stationary. There is $H' \supseteq H$ that is $\mathbb{P}$-generic over $V[G]$ – and also over $V[G_\gamma]$ – with $V[G_\gamma][H] \subseteq V[G_\gamma][H']$. In $V[G_\gamma][H]$, there is a function $F : \lambda^{<\omega} \to \lambda$ such that S does not meet the club C_F generated by F. However, because $\mathbb{P}$ is proper in $V[G]$, in $V[G][H']$ there is $s \in S$ such that $s \in C_F$, i.e., $\forall s' \subseteq_{<\omega} s : F(s') \in s$. But $s \in V[G_\gamma][H]$, hence in $V[G_\gamma][H]$, S indeed intersects C_F. Note that this argument works because we treat S, λ and F *as sets* (as opposed to something defined by a formula) which are naturally identical in each considered model.

Furthermore, if there were a $\mathbb{P}, \mathcal{D}$-generic filter in $V[G_\gamma]$ it would be in $V[G]$. So $\mathbb{P}, \mathcal{D}$ is a counterexample to PFA$^-$ in $V[G_\gamma]$. ω_1 is preserved along the iteration, so $\mathbb{P}$ retains it's cardinality ω_1 in $V[G_\gamma]$. Hence – there are no countable counterexamples to (B)PFA – $\mathbb{P}$ is a minimal counterexample, and hence there is some $q \leq p$ choosing $\mathbb{P}$ from the lottery sum in the γ-th step. By the Factor Lemma, q forces that there is a generic filter on $\mathbb{P}$. □

Corollary 4.31. *Theorem 4.30 implies that any LHMC iteration of* (B)PFA *with inaccessible length forces* PFA^-.

We conjecture that this result also gives some additional insight into the nature of the LHMC iterations of (B)PFA. In particular, we would hope that it would give some answers into what happens if an iteration stops (cf. Question 3.3), since we now know something about what counterexamples are (not) there at an inaccessible.

Inaccessibles seem to be the logical points where a LHMC iteration might stop. However, we were unable to achieve definite or interesting results.

4.7 $\mathrm{PFA}_{\mathfrak{c}}$ from a strongly unfoldable cardinal

This section reproduces [HJ09, Theorem 6] and the $\lambda = \kappa$ case of [Miy98, Theorem 3.1]. The original proof by Miyamoto first adds a Laver-like function via Easton support forcing [Miy98, Theorem 1.5] and then utilizes a Laver preparation, whereas Hamkins and Johnstone apply the proper lottery preparation with a fast-growing function that has what they call the *Menas property*, i.e., a function that anticipates hereditary sizes. We remove the need for a fast function altogether.

Hamkins and Johnstone state $\mathrm{PFA}_{\mathfrak{c}}$ for complete boolean algebras instead of arbitrary partial orders, but find $\mathcal{D}$-generic filters on $\mathbb{P}$. Miyamoto, like we do, considers arbitrary partial orders, but also finds a centered set instead of a filter.

We first define the appropriate large cardinal notion for the intended result. A preliminary notion is that of a κ*-model*:

Definition 4.32. *A transitive set M is called a* κ**-model** *iff* $M \models \mathrm{ZFC}^-$, $|M| = \kappa$, $M^{<\kappa} \subseteq M$ *and* $\kappa \in M$.

Definition 4.33. *An inaccessible cardinal κ is called* λ**-strongly unfoldable** *iff: For every κ-model M there is a transitive set N and an elementary embedding $j : M \to N$ with critical point κ, $j(\kappa) > \lambda$ and $V_\lambda \subseteq N$.*

A problem that arises here is that the κ-models may be too small to contain a counterexample $\mathbb{Q}$. Restricting ourselves to size κ, we can only find non-transitive models that contain names for large counterexamples. In the Baumgartner argument, we would want to consider a model X with $\mathbb{Q} \in X$, a $\mathbb{Q}$-generic filter H and use the object $X \cap H$, but if X is not transitive, it is not clear that $X \cap H$ would be centered w.r.t. $\mathbb{Q} \cap X$. We shall see that the techniques of special counterexamples, originally developed for the consistency proof of BPFA above, again give the required information about $\mathbb{Q}$ in a sufficiently small package.

Strongly unfoldable cardinals can clearly be compared to strong cardinals, however, they also exhibit behavior similar to supercompacts. Thus it is reasonable to assume that some version of Baumgartner's argument might work.

Fact 4.34. [HD06, Lemma 6] *If κ is δ-strongly unfoldable where δ is a limit and M is a κ-model, then there is a δ-strongly unfoldability embedding $j : M \to N$ with critical point κ, $j(\kappa) > \delta$, $N^{<\operatorname{cof}\delta} \subseteq N$ and $|N| = \beth_\delta$.*

Corollary 4.35. *In conclusion, if $\delta > \kappa$ is a limit, $M \in V_{\delta^+}$ is a κ-model and κ is δ^+-strongly unfoldable, there is an elementary embedding $j : M \to N$ such that:*

i. $\operatorname{crit}(j) = \kappa$, $j(\kappa) > \delta^+$,

ii. $V_{\delta^+} \subseteq N$,

iii. $N^\delta \subseteq N$,

iv. $|N| = \beth_{\delta^+}$, *and*

v. $M, j \in N$.

*We shall call such an embedding a δ^+-*strong unfoldability embedding.

Proof. v. is clearly implied by ii. and iii. □

Now we have collected the necessary notions to conduct the relative consistency proof.

Theorem 4.36. *If κ is strongly unfoldable, then the LHMC iteration of* PFA$_{\mathfrak{c}}$, *$\mathbb{P}_\kappa$, forces* PFA$_{\mathfrak{c}}$.

Proof. Suppose not. Let $p \in \mathbb{P}_\kappa$ such that p forces that $(\dot{D}, \dot{D}^*)$ is a special counterexample to $\mathrm{PFA}_{\mathfrak{c}}$. Let $\dot{\mathbb{Q}}$ be a name for a proper forcing witnessing this. Let G be a $\mathbb{P}_\kappa$-generic filter over V with $p \in G$. Note that $p \Vdash \mathfrak{c} \leq \kappa$ by Corollary 3.18. W.l.o.g. choose $\dot{D}$, $\dot{D}^* \subseteq H_\kappa$.

We need to get set up to utilize the strong unfoldability of κ, so we require a κ-model: Let $\lambda > \kappa$ be sufficiently large and regular such that

$$\mathbb{P}_\kappa, \dot{D}, \dot{D}^*, \dot{\mathbb{Q}} \in H_\lambda \text{ and } H_\lambda \models p \Vdash \Gamma^{\mathfrak{c}}(\dot{D}, \dot{D}^*),$$

and let $X \prec H_\lambda$ be a model with cardinality κ such that

$$\mathbb{P}_\kappa, \dot{D}, \dot{D}^*, \dot{\mathbb{Q}}, \kappa \in X,\ X^{<\kappa} \subseteq X \text{ and } H_\kappa = V_\kappa \subseteq X.$$

Notice the following: If $A \in X$ and $|A| \leq \kappa$ then $A \subseteq X$, since we can enumerate A in H_λ and find that enumeration in X by elementarity. In particular, this means $\mathrm{TC}(\mathbb{P}_\kappa) \subseteq X$ since $\mathrm{TC}(\mathbb{P}_\kappa)$ has size κ by Corollary 3.14. Alternatively, we could just assume $\mathbb{P}_\kappa \subseteq H_\kappa$ by Remark 3.15 and say $\mathrm{TC}(\mathbb{P}_\kappa) \subseteq H_\kappa \subseteq X$.

Let $\pi : X \to M$ be the Mostowski collapse of X. Then M is a κ-model. Now find some $\delta > \lambda$ such that $V_\delta[G]$ verifies $\mathbb{Q}$'s properness in $V[G]$ and let $j : M \to N$ be a δ^+-strong unfoldability embedding. Note that by Lemma 2.6 $\pi(\dot{D}) = \dot{D}$ and $\pi(\dot{D}^*) = \dot{D}^*$, i.e., $\dot{D}, \dot{D}^* \in M$, because their respective transitive closures are contained in X.

Likewise $\mathbb{P}_\kappa = \pi[\mathbb{P}_\kappa] \subseteq M$ and $\pi(\mathbb{P}_\kappa) = \mathbb{P}_\kappa \in M$ by Lemma 2.6. Therefore G is also $\mathbb{P}_\kappa$-generic over M. Work in $V[G]$. Now let H be $\mathbb{Q}$-generic over $V[G]$. Then for all $A \in D$, $H \cap A \cap M[G] \neq \emptyset$ because $A \subseteq M[G]$.

Because $\mathbb{Q} \in V_\lambda[G] \subseteq V_\delta[G] \subseteq N[G]$, $\mathbb{Q} \in N[G]$. By the choice of δ, $N[G]$ knows that $\mathbb{Q}$ is proper, a counterexample to $\mathrm{PFA}_{\mathfrak{c}}$ (because any D-generic filter in $N[G]$ would be in $V[G]$) and is hereditarily minimal with that property (because any smaller counterexample would also be a counterexample in $V[G]$). Also, $|\mathrm{TC}(\mathbb{Q})| < \lambda \leq j(\kappa)$.

$\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$ which in turn is an iteration of length $j(\kappa)$. Hence we can find some $q \leq p^\frown \mathbb{1}_{j(\kappa)}$ that chooses $\mathbb{Q}$ from the lottery sum in the κ-th step. Hence below q, $j(\mathbb{P}_\kappa)$ factors into $\mathbb{P}_\kappa * \mathbb{Q} * \mathbb{R}$. Let I be $\mathbb{R}$-generic over $V[G * H]$ and work now in $V[G * H * I]$. As in Theorem 4.6, j lifts to an embedding $j^* : M[G] \to N^* = N[G * H * I]$ by mapping $j^*(\sigma^G) = j(\sigma)^{G * H * I}$.

Recall that $j, M, H \in N^*$ and $D, D^* \in M[G]$. Consider $j^*[H \cap D^*] \in N^*$. We already know that for every $A \in D$ there is some $x_A \in H \cap M[G]$ such that $M[G] \models x_A \in A$, so by elementarity, $N^* \models j^*(x_A) \in j^*(A)$. Note that $j(D) = \{j(A) \mid A \in D\}$. Therefore $j^*[H \cap D^*]$ is a centered set on $j^*(\mathbb{Q})$ in $N[G * H * I]$; it intersects each of the $A \in j^*(D)$, thus it is $(j^*(\mathbb{Q}), j^*(D))$-generic. Hence, by elementarity, there is a $(\mathbb{Q}, D)$-generic centered set h in $M[G]$.

By Lemma 3.28, $h \cap D^*$ can be extended to a centered set on *any* witness to $\Gamma^{\mathfrak{c}}(D, D^*)$. Thus $M[G] \models \neg\Gamma^{\mathfrak{c}}(D, D^*)$. □

Since PFA_ω is trivial (a special counterexample would be countable), we need to enhance the iteration to get an interesting result.

Corollary 4.37. *The* adding *counterexamples to* $\mathrm{PFA}_\mathfrak{c}$ *iteration forces* $\mathrm{PFA}_\mathfrak{c}$ *with* $\mathfrak{c} > \aleph_1$. *We can also make the iteration collapsing to force* $\mathfrak{c} = \aleph_2$.

In a previous version of this proof, we would do most of the work with the non-transitive model X instead of M. As hinted above, one runs into the problem that $X[G] \cap H$ might not generate a $\mathbb{Q}$-centered set on X. This issue can be resolved by computing a set D^* as in Lemma 3.29 and considering $X[G] \cap H \cap D^*$.

This would be somewhat closer to the original argument by Hamkins and Johnstone. However, using such a D^* requires some technical bulk, so the proof we ended up with seems preferable. Also the general tediousness of working with non-transitive models is kept to a minimum.

4.8 AAFA($\mathfrak{c}$) from a weakly compact cardinal

In the previous section we have shown that Baumgartner's classical argument for PFA (cf. Section 4.1) applies to strongly unfoldable cardinals in addition to the (much stronger) supercompacts. We shall now show that the basic techniques of the argument are even applicable to weakly compact cardinals.

We reproduce a result by Baumgartner, stating that the fragment of PFA that only considers Axiom A forcings of size $\leq 2^\omega$ is consistent relative to the existence of a weakly compact cardinal, cf. [Bau84, Theorem 9.2]. Baumgartner states this theorem as an unpublished result of his and a proof seems to be unavailable (or lost) in the available literature. We conduct a novel proof via a LHMC iteration.

We recall what it means for a forcing notion to satisfy Axiom A. The treatment of Axiom A in [Wei08] is the most useful one for our proof.

Definition 4.38. [Wei08, 2.18] *A forcing notion $\mathbb{P}$ is satisfies* **Axiom A** *iff there is a set $(\leq_n)_{n\in\omega}$ of partial orders on $\mathbb{P}$ such that*

i. $p \leq_0 q \rightarrow p \leq q$ *and* $p \leq_{n+1} q \rightarrow p \leq_n q$ *for all* $n \in \omega$,

ii. *if* $(p_n)_{n\in\omega}$ *is a sequence with* $p_0 \geq_0 p_1 \geq_1 p_2 \dots$ *then there is a* q *such that* $q \leq_n p_n$ *for all* n, *and*

iii. *If* $p \in \mathbb{P}$, $A \subseteq \mathbb{P}$ *is a maximal antichain below* p *and* $n < \omega$, *then there is a* $q \leq_n p$ *such that* $|\{a \mid a \in A \wedge a \text{ and } q \text{ are compatible}\}| < \aleph_1$.

Now we can state the axiom we are interested in.

Axiom 4.39 (Axiom A Forcing Axiom)**.** AAFA *is the restriction of* PFA *to Axiom A forcings.* AAFA($\mathfrak{c}$) *is the restriction of* AAFA *to forcings of size* $\leq 2^\omega$.

To make sense of Axiom A in a LHMC iteration we require the following fact, provable via the *proper game*.

Fact 4.40. [Jec03, 31.11] *If $\mathbb{P}$ is an Axiom A forcing notion, then $\mathbb{P}$ is proper.*

We also choose a notion of weak compactness that is conducive to our proof. The following characterization of weakly compact cardinals (there as Π^1_1-indescribables) is implicit in [Hau91, Theorem 1.3] and made explicit in [Ham02, Lemma 17.1]. Calling it the "Hauser property" follows the intent of [HJ09].

Definition 4.41. *A cardinal κ with $\kappa^{<\kappa} = \kappa$ is* **weakly compact** *iff it satisfies the* Hauser property*: for each κ-model M there is a transitive N with $N^{<\kappa} \subseteq N$ and an elementary embedding $j : M \rightarrow N$, $\mathrm{crit}(j) = \kappa$ with $j, M \in N$.*

We shall now see that this characterization is indeed perfectly suited for a LHMC iteration.

Theorem 4.42. *If κ is weakly compact, then the adding collapsing LHMC iteration of* AAFA($\mathfrak{c}$), $\mathbb{P}_\kappa$, *forces* AAFA($\mathfrak{c}$) *with* $\mathfrak{c} = \aleph_2$.

Proof. Suppose the theorem is false. Let $p \in \mathbb{P}_\kappa$ such that p forces that $(\dot{\mathbb{Q}}, \dot{\mathcal{D}})$ is a hereditarily minimal counterexample to AAFA($\mathfrak{c}$) Let $\dot{\mathcal{A}}$ be a name for a sequence of partial orders on $\dot{\mathbb{Q}}$ witnessing Axiom A. Let G be a generic filter with $p \in G$.

Note that by Corollary 3.21, $p \Vdash \mathfrak{c} = \kappa = \aleph_2$, so we can w.l.o.g. assume that $p \Vdash \dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq \kappa$. Thus by Lemma 2.9 we can suppose that $\dot{\mathbb{Q}}, \dot{\mathcal{A}} \subseteq H_\kappa$. Let λ be large enough such that H_λ knows that $\dot{\mathbb{Q}}$ is a name for an Axiom A forcing as witnessed by $\dot{\mathcal{A}}$. Let $X \prec H_\lambda$ with $H_\kappa \subseteq X$, $\mathbb{P}_\kappa, \dot{\mathbb{Q}}, \dot{\mathcal{D}}, \dot{\mathcal{A}}, \kappa \in X$, $X^{<\kappa} \subseteq X$ and $|X| = \kappa$.

Let $X \to M$ be the Mostowski collapse of X, then M is a κ-model. Notice that since $\dot{\mathbb{Q}} \subseteq H_\kappa \subseteq X$ is in the transitive part of X, $\pi(\dot{\mathbb{Q}}) = \dot{\mathbb{Q}} \in M$. Likewise $\dot{\mathcal{D}} \in M$, $\dot{\mathcal{A}} \in M$ and $\mathbb{P}_\kappa \in M$. Now let $j : M \to N$ be a κ-weakly compactness embedding.

Now work in $V[G]$. Note that $M[G] \in N[G]$ by the Hauser property and since $N[G]$ is transitive it contains all the sets we require (since we put them in $M[G]$). Check that $\mathcal{A} = (\leq_n)_{n\in\omega} \in M[G] \in N[G]$ also witnesses that $\mathbb{Q}$ satisfies Axiom A in $N[G]$. *i.* and *ii.* from Definition 4.38 are clear. Now check *iii.*: Let p, n, A be required. Then, in $V[G]$, there is some $q \leq_n p$ such that $|\{a \mid a \in A \wedge a \text{ and } q \text{ are compatible}\}| < \aleph_1$. Since $N[G]$ and $V[G]$ agree on $\aleph_1$ (since $\mathbb{P}_\kappa$ is proper) and on the computation of that set (since $\mathbb{Q} \subseteq N[G]$), this is also true in $N[G]$. By the usual arguments, $N[G]$ then also knows that $\mathbb{Q}$ is a hereditarily minimal counterexample to AAFA($\mathfrak{c}$). Also $|\mathrm{TC}(\mathbb{Q})| < \kappa^+ \leq j(\kappa)$.

Now consider $j(\mathbb{P}_\kappa)$. $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$ which in turn is an iteration of length $j(\kappa)$. Hence we can find some $q \leq p^\frown \mathbb{1}_{j(\kappa)}$ that chooses $\mathbb{Q}$ from the lottery sum in the κ-th step, i.e., below q, $j(\mathbb{P}_\kappa)$ factors into $\mathbb{P}_\kappa * \mathbb{Q} * \mathbb{R}$. Let H be $\mathbb{Q}$-generic over $V[G]$ and I be $\mathbb{R}$-generic over $V[G * H]$. Now work in $V[G * H * I]$. As in Theorem 4.6, j lifts to an embedding $j^* : M[G] \to N^* = N[G * H * I]$ by mapping $j^*(\sigma^G) = j(\sigma)^{G*H*I}$.

Because $j^*, H \in N^*$, the set $j^*[H]$ is an element of N^* and generates a filter on $j^*(\mathbb{Q})$. Because H is $\mathbb{P}$-generic over M, for each $D \in \mathcal{D}$, there is some $x_D \in D \cap H$. Hence, by elementarity, $N^* \models j^*(x_D) \in j^*(D)$. Thus the filter generated by $j^*[H]$ is $(j^*(\mathbb{Q}), j^*(\mathcal{D}))$-generic. Again by elementarity, there must be a $(\mathbb{Q}, \mathcal{D})$-generic filter in $M[G]$. This filter would also be in $V[G]$ and contradict that $\mathbb{Q}$ is a counterexample to AAFA($\mathfrak{c}$). □

This answers (more or less) a question posed by Johnstone in 2009 [Joh09] about the proper lottery preparation.

Question 4.43 (Johnstone)**.** *Which fragment of* PFA *can we get from a weakly compact cardinal?*

4.9 PFA($\mathfrak{c}$) from a Σ^2_1-indescribable cardinal

We were hoping to improve on the previous result, extending Baumgartner's classical result on weakly compact cardinals to PFA($\mathfrak{c}$). We later learned that this is in fact a known open question in this area of research [NS08, below Corollary 6.].

Despite many efforts, we were only able to bound the consistency strength of PFA($\mathfrak{c}$) to a Σ^2_1-indescribable. The primary problem is a sufficiently "small" characterization of properness. In the end, we resolved to reproduce this upper bound found by Neeman and Schimmerling [NS08].

Axiom 4.44. PFA($\mathfrak{c}$) *is the restriction of* PFA *to forcings of size* $\leq 2^\omega$.

We shall again choose a beneficial characterization of the large cardinal notion we use.

Definition 4.45. *A cardinal* κ *is* **Σ^2_1-indescribable** *iff: For all* $Q \subseteq H_\kappa$ *and formulae* φ*,* **if** *there is* $F \subseteq H_{\kappa^+}$ *such that* $(H_{\kappa^+}, F) \models \varphi(Q)$*,* **then** *there is* $\lambda < \kappa$ *such that* $(H_{\lambda^+}, F \cap H_{\lambda^+}) \models \varphi(Q \cap H_\lambda)$ **and** $(H_\lambda, Q \cap H_\lambda) \prec (H_\kappa, Q)$*.*

Up to the last condition this is the usual definition of indescribability, see, e.g., [Kan09, p. 58]. Neeman and Schimmerling showed that we can add the elementary submodel condition in [NS08, Lemma 8].

We will show that this characterization is well adapted to the needs of a LHMC iteration. This is not surprising, since we adapted the definition from a more general result that uses the *universal iteration* by Neeman and Schimmerling.

Theorem 4.46. *If* κ *is* Σ^2_1*-indescribable, then the adding collapsing LHMC iteration of* PFA($\mathfrak{c}$)*,* $\mathbb{P}_\kappa$*, forces* PFA($\mathfrak{c}$)*.*

Proof. This proof works like a downwards version of Baumgartner's argument. Instead of extending the LHMC iteration, we find a suitable way to factor it into smaller forcings. We then lift the elementarity provided by indescribability. Aside from going downwards instead of upwards it does not differ that much from the proof of, e.g., Theorem 4.42.

Suppose the theorem is false. Let G be some generic filter on $\mathbb{P}_\kappa$ and briefly work in $V[G]$. First note that by Corollary 3.21, $\mathfrak{c} = 2^{\omega_1} = \kappa = \aleph_2$. There is a hereditarily minimal counterexample to PFA($\mathfrak{c}$) $\mathbb{Q}, \mathcal{D} \subseteq H_\kappa$[4] and some club $C \subseteq [H_{\kappa^+}]^\omega$ witnessing that $\mathbb{Q}$ is proper. Furthermore, there a function $f : [H_{\kappa^+}]^{<\omega} \to H_{\kappa^+}$ with $C_f \subseteq C$. Clearly, C_f also witnesses that $\dot{\mathbb{Q}}^G$ is proper.

Work in V again. Let $\dot{\mathbb{Q}}$, $\dot{\mathcal{D}}$ and $\dot{f}$ be names for the counterexample and the properness-witnessing function above. By Lemma 2.9 we can w.l.o.g. assume that $\dot{\mathbb{Q}}, \dot{\mathcal{D}} \subseteq H_\kappa$ and $\dot{f} \subseteq H_{\kappa^+}$. Recall that by Corollary 3.14, $\mathbb{P}_\kappa \in H_{\kappa^+}$.

We use the appropriate variant of Lemma 3.16 for the following consideration. Let $\varphi(X)$ be a formula in the language $\{\in, F\}$ saying:

> "$X = (Q, D, P, \eta)$ where Q, D are P-names for a hereditarily minimal counterexample to PFA($\mathfrak{c}$) where Q's properness it witnessed by the P-name for a function F, η is inaccessible and P is the LHMC iteration of PFA($\mathfrak{c}$) as defined in H_η."

Using Remark 3.15 we can code[5] $(\dot{\mathbb{Q}}, \dot{\mathcal{D}}, \mathbb{P}_\kappa, \kappa)$ into some $X \subseteq H_\kappa$. W.l.o.g. just say that X equals that 4-tuple.

We shall now verify that $(H_{\kappa^+}, \dot{f}) \models \varphi(X)$. Let G be a H_{κ^+}-generic filter on $\mathbb{P}_\kappa$. H_{κ^+} contains $\mathbb{P}_\kappa$ and all (dense) subsets of $\mathbb{P}_\kappa$, so H_{κ^+} and V agree on genericity, i.e., G is also V-generic. Work in $V[G]$. Note that we write $H_{\kappa^+} = H_{\kappa^+}^{V[G]} = H_{\kappa^+}[G]$. $(\mathbb{Q}, \mathcal{D}) := (\dot{\mathbb{Q}}^G, \dot{\mathcal{D}}^G)$ is a counterexample to PFA($\mathfrak{c}$) = PFA(κ). $(H_{\kappa^+}, \dot{f}^G)$ knows that $\mathbb{Q}$ is proper, and if there would be a $(\mathbb{Q}, \mathcal{D})$-generic filter in H_{κ^+}, it would be in $V[G]$, contradicting that $(\mathbb{Q}, \mathcal{D})$ is a counterexample there.

Finally, if $\mathbb{Q}' \in H_{\kappa^+}$ is a hereditarily *smaller* counterexample than $\mathbb{Q}$, $\mathbb{Q}'$ has hereditary size smaller κ (i.e. ω_1), so $\mathbb{Q}' \in H_{\omega_2}$. We shall now show that $\mathbb{Q}$ is really proper by a variant of the argument of Lemma 2.22. Notice that $|[H_{\omega_2}]^\omega| = |H_{\omega_2}| \leq 2^{<\omega_2} = 2^{\omega_1} = \aleph_2 < \aleph_3$. So $[H_{\omega_2}]^\omega \in H_{\kappa^+} = H_{\aleph_3}$. Hence clubs in $[H_{\omega_2}]^\omega$ are absolute between H_{κ^+} and $V[G]$, so the characterization of properness in Lemma 2.20 is absolute. Thus $\mathbb{Q}'$ is proper in $V[G]$, i.e., a counterexample there, contradicting $\mathbb{Q}$'s minimality.

[4] One may want to code the sequence $\mathcal{D}$ into an actual subset of H_κ here.

[5] e.g. Gödel Pairing.

Work in V again. Now find an appropriate $\lambda < \kappa$ by Σ^2_1-indescribability, i.e., $(H_{\lambda^+}, \dot{f}') \models \varphi(X')$ where $f' = \dot{f} \cap H_{\lambda^+}$ and

$$X' = (\dot{\mathbb{Q}}, \dot{\mathcal{D}}, \mathbb{P}_\kappa, \kappa) \cap H_\lambda = (\dot{\mathbb{Q}} \cap H_\lambda, \dot{\mathcal{D}} \cap H_\lambda, \mathbb{P}_\kappa \cap H_\lambda, \lambda)$$
$$= (\dot{\mathbb{Q}} \cap H_\lambda, \dot{\mathcal{D}} \cap H_\lambda, \mathbb{P}_\lambda, \lambda).$$

The last equality holds because by the definition of φ, $\mathbb{P}_\kappa \cap H_\lambda$ is the LHMC iteration of PFA($\mathfrak{c}$) as defined in H_λ, i.e., $\mathbb{P}_\lambda$. Now, $\dot{\mathbb{Q}} \cap H_\lambda$ appears in the lottery sum of $\mathbb{P}_\kappa$ in the λ-th step. Find a condition $p \in \mathbb{P}_\kappa$ selecting $\dot{\mathbb{Q}} \cap H_\lambda$.

Because $H_\lambda \prec H_\kappa$, the definitions of the stages of the iterations at $\alpha < \lambda$ in $\mathbb{P}_\lambda$ and $\mathbb{P}_\kappa$ are the same. Thus, below p, the longer iteration $\mathbb{P}_\kappa$ factors into $\mathbb{P}_\lambda * (\dot{\mathbb{Q}} \cap H_\lambda) * \mathbb{P}_{\lambda+1,\kappa}$. Let $G * H * R$ be a generic filter for that factorization. Note that we force over V and never, despite $\mathbb{P}_\kappa$ being a class in H_κ, use anything like class forcing.

Now work in $V[G * H * R]$. As usual (cf. Theorem 4.6), we can lift[6] the elementarity to

$$(H_\lambda[G], f') \prec (H_\kappa[G * H * R], f).$$

Let $\mathbb{Q} := \dot{\mathbb{Q}}^{G*H*R}$ and $\mathcal{D} := \dot{\mathcal{D}}^{G*H*R}$. It is trivial that $(\dot{\mathbb{Q}} \cap H_\lambda)^G \subseteq \mathbb{Q}$. The usual argument shows that for each $A \in \mathcal{D}$ there is $A' \in (\dot{\mathcal{D}} \cap H_\lambda)^G$ with $A' \subseteq A$. Note that $H \in H_\kappa[G * H * R]$ and (by elementarity) H extends to a filter H' on $\mathbb{Q}$. By genericity, H intersects every $A' \in (\mathcal{D} \cap H_\lambda)^G$, so H' is $(\mathbb{Q}, \mathcal{D})$-generic. Then $\mathbb{Q}$ could not have been a counterexample to PFA($\mathfrak{c}$) in the first place. □

This proof is a loose yet faithful adaption of the proof of [NS08, Theorem 12]. Our proof merely drops the requirement of a fast function and adapts to that slightly changed situation. Aside from that, we kept true to their methods and so we strongly believe that a LHMC iteration would prove the full result just as well.

However, the LHMC iteration is at a disadvantage to the universal iteration. For the universal iteration $\mathbb{I}_\kappa$ it holds that for $\lambda < \kappa$ $\mathbb{I}_\kappa \cap H_\lambda$ is an initial segment of $\mathbb{I}_\kappa$. This is a priori not true for LHMC iterations, as in some stage below λ, the minimal counterexamples may already lie above H_λ. The LHMC iteration of length λ would then stop whereas the LHMC

[6] $\sigma^G \mapsto \sigma^{G*H*R}$ is easily shown to be the identity map.

iteration of length $\kappa > \lambda$ might continue. This seems related to Question 3.3. Nevertheless, this issue does not arise here as the elementarity provided by Σ^2_1-indescribables is sufficient to avoid this problem.

One would think that an argument similar to the one in Section 4.8 would also work here, using a Hauser-characterization of indescribability. The *Hauser property*, $j, M \in N$ for elementary $j \in M \to N$, seems like the optimal way to conduct a LHMC iteration. However, we were not able to make it work for a weakly compact cardinal. For completeness sake, we cite the appropriate characterization. The following theorem is due to Hauser [Hau91, Theorem 1.3], [Hau92, p. 381] and its formulation is adapted from [Ham02, p. 4].

Theorem 4.47. *Let $m, n \geq 1$. An inaccessible cardinal κ is Π^m_n-indescribable iff for every κ-model M there is a transitive N and an elementary embedding $j : M \to N$ such that $j, M \in N$, $\operatorname{crit}(j) = \kappa$ and N is Σ^m_{n-1}-correct for κ, i.e., $(V_{\kappa+m})^N \prec_{\Sigma_{n-1}} V_{\kappa+m}$ and $N^{|V_{\kappa+m-2}|} \subseteq N$ ($N^{<\kappa} \subseteq N$ for $m = 1$).*

If we attempt the consistency proof of PFA($\mathfrak{c}$) from a *weakly compact cardinal* as in Theorem 4.42, we run into the problem that we are seemingly unable to verify that $\mathbb{Q}$ is actually proper in $N[G]$. The witnesses to properness provided by our best characterizations were too large resp. too complex to verify this in the appropriate $V_{\kappa+m}$. Compare with Axiom A where the witnesses (sequences of orderings) have the same size as the forcing itself. Nevertheless, we do think that it is entirely plausible that some such argument would work.

Conjecture 4.48. *A Π^1_1-indescribable, i.e., a weakly compact, cardinal is sufficient.*

4.10 RA(proper) from an uplifting cardinal

Lastly, we provide an example where we apply our method to an axiom outside the realm of PFA. We consider a novel axiom by Hamkins and Johnstone. This section also serves to illustrate further how LHMC iterations adapt to proofs using proper lottery preparations. All definitions and results in this section are due to Johnstone [Joh10]; he proved the main theorem using a proper lottery preparation.

We replicate Johnstone's proof using a LHMC iteration. The proof fits in the scheme of our previous arguments, since we again find some appropriate way to "lengthen" the LHMC iteration and finding what we need in the larger generic extension.

Axiom 4.49 (Resurrection Axiom (RA))**.** RA(proper) *is the following axiom: For each proper notion of forcing* $\mathbb{P}$*, there is a* $\mathbb{P}$*-name* $\dot{\mathbb{Q}}$ *for a proper forcing such that whenever* $G * \dot{H}$ *is* $\mathbb{P} * \dot{\mathbb{Q}}$*-generic, then* $H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V[G*\dot{H}]}$.

The following large cardinal notion was purposefully invented to fit RA(proper)[7].

Definition 4.50. *A regular cardinal* κ *is called* **uplifting** *iff there are arbitrarily large regular* γ *with* $V_\kappa \prec V_\gamma$*.*

We show that uplifting cardinals are indeed large, so we can apply our niceness criteria to a LHMC iteration of uplifting length.

Lemma 4.51. *If* κ *is uplifting and* $V_\kappa \prec V_\lambda$*, then* κ *and* λ *are inaccessible.*

Proof. Let $\alpha < \kappa$ and $2^\alpha \geq \kappa$. Then for all $x \in 2^\alpha$, $x \in V_\kappa$. Note that $V_\kappa \models \forall x : x \neq 2^\alpha$. There is some $\gamma > 2^\alpha$ with $V_\kappa \prec V_\gamma$ since κ is uplifting. But $V_\gamma \models \exists x : x = 2^\alpha$. Contradiction. Hence κ is inaccessible.

V_λ also knows all $x \in 2^\alpha$ for any $\alpha < \lambda$ and because κ is inaccessible, by elementarity $V_\lambda \models \forall \alpha \exists x : x = 2^\alpha$. So λ is inaccessible. □

Now that we have established that they are large, the place of upliftings in the large cardinal hierarchy is the next natural question.

Remark 4.52. *In consistency strength, uplifting cardinals are stronger than reflecting cardinals, but weaker than Mahlo cardinals.*

Proof. The proof of Remark 4.12 shows that if μ is Mahlo, then V_μ has uplifting cardinals. Also, if κ is uplifting, it is reflecting: Suppose φ is a formula, $a \in H_\kappa$ and for some $\delta \geq \kappa$, $H_\delta \models \varphi(a)$. Then there is some sufficiently large $\gamma > \delta$ with $H_\kappa \prec H_\gamma$. And since $H_\gamma \models$ "$\exists \delta : H_\delta \models \varphi(a)$", so does H_κ, i.e., there is $\delta' < \kappa$ with $H_{\delta'} \models \varphi(a)$. □

Finally we show the relative consistency of the proper Resurrection Axiom.

Theorem 4.53. *Let* κ *be an uplifting cardinal. The adding LHMC iteration of* RA(proper)*,* $\mathbb{P}_\kappa$*, of length* κ *forces* RA(proper)*.*

[7] As T. Johnstone told me in a personal conversation.

Proof. Assume there is $p \in \mathbb{P}_\kappa$ and a name for a proper forcing $\dot{\mathbb{Q}}$ such that p forces the failure of RA(proper) on $\dot{\mathbb{Q}}$. W.l.o.g. let p force that $\dot{\mathbb{Q}}$ is of minimal hereditary size with the failure of RA(proper). Find some uplifting $\gamma > \kappa$ with $H_\kappa \prec H_\gamma$ and $p \Vdash \dot{\mathbb{Q}} \in H_\gamma$.

Note that as in Lemma 3.16 we can w.l.o.g. assume that $\mathbb{P}_\kappa$ is a definable class in H_κ[8]. To see this we only need to know that for names $\dot{P} \in H_\kappa$ the property "being a name for a hereditarily minimal counterexample to RA(proper)" is H_κ-V-absolute. Assume $H_\kappa \models$ "$\dot{P}$ contradicts RA(proper)" and $\dot{P}$ is not really a name for a counterexample to RA(proper). Then there is a name $\dot{Q}$ for a $\dot{P}$-name witnessing RA(proper). This name may lie outside of H_κ, but there will be an appropriate uplifting $\delta > \kappa$ that knows that $\dot{Q}$ witnesses RA(proper) on $\dot{P}$. By elementarity there must be such a name in H_κ. The reverse direction is trivial and everything else can be done as in Lemma 3.16.

Let $\varphi(x)$ define $\mathbb{P}_\kappa$ in H_κ. Hence if we evaluate φ in H_γ, we obtain a respective definable class $\mathbb{P}^*_\gamma$ in H_γ. As usual, sufficient elementarity gives us that $\mathbb{P}^*_\gamma$ is a proper extension of $\mathbb{P}_\kappa$, so $\mathbb{P}^*_\gamma$ factors into $\mathbb{P} * \dot{\mathbb{Q}}' * \dot{\mathbb{R}}$ where $\dot{\mathbb{Q}}'$ is a name for the lottery sum of all minimal counterexamples to RA(proper). Hence we may find $q \leq p^\frown \mathbb{1}_\gamma$ that chooses $\dot{\mathbb{Q}}$ in the κ-th step.

Let G, H, I be respectively $\mathbb{P}_\kappa$, $\mathbb{Q}$, $\mathbb{R}$-generic. Note that because $\mathbb{P}_\kappa$ and $\mathbb{P}^*_\gamma$ are defined by the same formula, $H_\kappa[G] \prec H_\gamma[G * H * I]$. By Corollary 3.19, $\mathbb{P}_\kappa$ forces that $\kappa = \mathfrak{c}$ and $H_\mathfrak{c}^{V[G]} = H_\kappa[G]$, and $\mathbb{P}^*_\gamma$ forces the same for γ.

Now conclude $H_\mathfrak{c}^{V[G]} = H_\kappa[G] \prec H_\gamma[G * H * I] = H_\mathfrak{c}^{V[G*H*I]}$. Hence $\mathbb{Q}$ satisfies RA(proper) as witnessed by $\dot{\mathbb{R}}$ which is proper as a countable support iteration of proper forcings. □

[8] This is not really important. We can just as well define $\mathbb{P}_\kappa$ to be a LHMC iteration *inside* H_κ without caring whether or not this is the actual LHMC iteration. Elementarity would automatically take care of everything else by the same arguments.

5 Further considerations

5.1 Streamlining the approach

We hoped that we could describe *one* iteration that, given a suitable large cardinal, would force the various fragments of PFA – and related forcing axioms – with no further modifications. The interest in finding such an unified approach lies in the mapping of large cardinal axioms to forcing axioms (in consistency strength). In the current state of research, whenever an axiom is proposed, the consistency proof is a novel effort, and does not easily fit the pattern of previous such arguments. A notable exception is the proof of SPFA/MM which follows the one for PFA quite closely.

With a unified approach, one might just put in the desired consistency strength (i.e. a large cardinal) in a largely pre-made framework and produce a corresponding forcing axiom with reduced effort. In opposite direction, one may consider the LHMC iteration of whatever axiom one fancies at the time and engineer from that a suitable large cardinal to apply that same pre-made framework. To our best understanding, this is what gave rise to *uplifting cardinals* in the context of RA(proper), as treated in Section 4.10.

The main reason for using axiom-specific iterations is that in all arguments we need to find our supposed counterexample in the lottery sum at a specific stage of the iteration. For example, we were unable to force BPFA with the LHMC iteration of PFA, since there might be a small counterexample to PFA (with large dense sets) and the smallest counterexample to BPFA is a bigger forcing (but with small antichains). In this case, the iteration does not cover the counterexample to BPFA.

Examining the proofs in this thesis, one might find that similar looking axioms have quite comparable consistency proofs within the framework of LHMC iterations. Though we failed to achieve a completely unified iteration, we shall now further streamline the methods on certain, similar axioms. In particular we differentiate between cardinality-wise fragments and bounded fragments of PFA.

On the one hand, whenever we consider a fragment of PFA where we restrict by cardinality, the LHMC iteration of PFA itself just works. This includes Sections 4.1, 4.5, 4.6 and 4.9. It is also worth noting that the arguments for PFA and for PFA($\mathfrak{c}$) are very similar in approach; the crucial step in both proofs is the continuation of the forcing and the lift of the elementary embedding. We shall call this approach the *Baumgartner argument* due to its similarity with Baumgartner's original proof of Corollary 4.7.

The proofs in 4.5 and 4.6 are also nearly identical. If this basic technique should be deserving of a name, *pulldown argument* would be quite appropriate.

The other typical fragments of PFA are the *bounded* fragments that restrict not the cardinality of the forcing itself, but of the predense sets. Since these axioms do not control the cardinality of their subject forcings, these are harder to approach with our method. We propose the following "unified" iteration for these axioms.

Definition 5.1. *The* **bounded LHMC iteration of PFA**[9] **with length** κ *is the countable support iteration of* $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa)$*, where* $\mathbb{P}_\alpha$ *and* $\dot{\mathbb{Q}}_\alpha$ *are defined by induction:*

Given α*, let* $\dot{\nu}$ *be a hereditarily minimal* $\mathbb{P}_\alpha$*-name for the smallest cardinal such that* $\mathrm{PFA}_{\dot{\nu}}$ *fails, i.e.,* $\mathbb{1}_\alpha \Vdash$ "$\mathrm{PFA}_{\dot{\nu}}$ *fails and* $\forall \nu' < \dot{\nu} : \mathrm{PFA}_{\nu'}$"*. Let* $\dot{\mu}$ *be a* $\mathbb{P}_\alpha$*-name for the hereditary size of a hereditarily minimal counterexample to* $\mathrm{PFA}_{\dot{\nu}}$*. Let* $\dot{\mathbb{Q}}_\alpha$ *be a* $\mathbb{P}_\alpha$*-name for the lottery sum of* all proper forcings *with hereditary size* $\leq \dot{\mu}$

We want to force with *all* proper forcings of a certain size (instead of counterexamples only) because there might be some pathological cases where there is a small forcings with big antichains (while the counterexamples we found are big forcings with small antichains). But we need to capture that small forcing despite minimizing for the size of the antichains first. Theorem 5.5 below illustrates this issue.

One might imagine a similar iteration by fragmenting PFA into PFA(λ), viz., restricting PFA by the cardinality of the forcing, not the size of the $D \in \mathcal{D}$. One could then find in each stage of the iteration a name for the smallest λ where PFA(λ) fails and continue as in the *bounded* LHMC iteration. However, this would not differ much (cf. Remark 3.5) from the usual LHMC iteration of PFA, as whenever PFA(λ) fails there is a counterexample to PFA of hereditary size λ and – naturally – vice versa. So by finding the smallest λ where PFA there is a counterexample to PFA in H_{λ^+} is the same as finding the smallest λ where PFA(λ) fails.

[9] We would like to call this the LHMC iteration for bounded PFA, but this would sound like the LHMC iteration of BPFA.

With that in mind, we can say that the LHMC iteration of PFA is indeed the "unified" iteration for cardinality-wise fragments of PFA. Therefore we conclude that to arrive at a unified method for some class of forcing axioms – or just fragments of PFA – one should search for a suitable way to order them in a way that can be minimized and construct some such iteration.

Nevertheless, the arguments for BPFA and $PFA_{\mathfrak{c}}$ are again quite similar; the crucial observation is the fact that the technique of *special counterexamples* codes enough information about the considered forcing notions to conduct the consistency proof. We are inclined to call this the *bounded argument*, despite Section 4.7 being actually inspired by the Baumgartner argument and Section 4.3 using the pulldown argument.

5.2 Hierarchies of forcing axioms

Lemma 4.6 gives the classical hierarchy of cardinality-wise fragments of PFA, bounding their consistency strengths by fragments of supercompact cardinals. Miyamoto [Miy98, 3.1] gives a related hierarchy of either cardinality-wise fragments [Miy98, 2.2] or bounded fragments where the condition to find a filter is weakened to a centered set [Miy98, 2.5]. The interest of such hierarchies lies in the investigation of the consistency strength of certain natural principles.

As forcing is a mere technique, an artificial construct made up by mathematicians, forcing axioms are bad candidates for new axioms if any semblance of naturalness is to be preserved. Sure, modern mathematics has removed itself from the original intent, and claim to naturalness, of Euclid and his contemporaries. But as the continued application of mathematics to real-world circumstances indicate, the principles on which even modern mathematics is founded may be elective, but they cannot be truly *arbitrary*.

This is not to say these axioms are without *merit*. Just as Zorn's Lemma – a quite unnatural concept, seldom considered an axiom – is just a version of the Axiom of Choice to apply without a deeper understanding of set theory, the forcing axioms are an easy access to combinatorial principles with high consistency strength that do not require a deeper dive into models of set theory.

Whatever their claim to naturalness may be, these axioms solve many interesting – and, indeed, natural – questions, usually of a combinatorial nature. The prime example here is obviously the Continuum Hypothesis, resolved by PFA as $\mathfrak{c} = \aleph_2$ [Jec03, Theorem 31.23].

However, if providing strong enough principles to resolve any interesting question were our only ambition, we might just as well assume $0 = 1$ and be done with it[10]. Assuming $0 = 1$ is obviously nonsense, so we resolve to measure the consistency strength of whatever natural questions we care about as precisely as possible. Keeping with the Continuum Hypothesis as an example, this is already resolved by BPFA [Moo05] – a much weaker principle than PFA.

Deconstructing a proof of any such resolution to a natural question may yield the information that only a certain kind of forcing notion covered by the axiom is required. Given some known fragmentation/hierarchy of that forcing axiom would immediately yield a new lower bound in consistency strength.

Sections 4.7 and 4.8 give immediate rise to certain large cardinal notions and associated hierarchies of fragments of PFA. First, we generalize the notions of *unfoldable cardinal* and *weakly compact cardinal* to encompass larger models. This allows us to code more information in these models and apply the Baumgartner argument to them. The generalization of unfoldables follows the definition of strong unfoldability with the Hauser property.

Definition 5.2. *A cardinal κ is* **strongly unfoldable on** λ *for $\lambda \geq \kappa$ iff for all μ: For each λ-model M there is an elementary embedding $j : M \to N$ with critical point κ, $j(\kappa) > \mu$, $V_\mu \subseteq N$ and $j, M \in N$.*

As for the generalization of weakly compacts, we use the characterization of Definition 4.41.

Definition 5.3. *A cardinal κ with $\kappa^{<\kappa} = \kappa$ is* **λ-weakly compact** *for $\lambda \geq \kappa$ iff for each λ-model M there is a transitive N with $N^{<\lambda} \subseteq N$ and an elementary embedding $j : M \to N$, $\mathrm{crit}(j) = \kappa$ with $j, M \in N$.*

Note that κ is strongly unfoldable iff it is strongly unfoldable on κ, and weakly compact iff it is κ-weakly compact. We conjecture that cardinals that are strongly unfoldable on λ or λ-weakly compact for all λ are in fact supercompact, or at least equiconsistent to them, since they can be used to force AAFA or PFA (see below).

[10] I'll have to admit here that I borrowed this slightly absurd line of thought from Grigor Sargsyan's justification for Inner Model Theory he gave in a lecture at UC Irvine, 2012.

Also, Miyamoto defines a related large cardinal notion; the H_λ-reflecting cardinals. [Miy98, Theorem 1.4] suggests that a cardinal is strongly unfoldable on λ iff it is H_{λ^+}-reflecting. However, the Theorem includes Laver-like function which we do not require. Nevertheless, this result would imply that a cardinal which is strongly unfoldable on λ for all λ is in fact supercompact by [Miy98, Proposition 1.2].

Now the proofs of Theorems 4.36 and 4.42 generalize to the following hierarchies. Note that the proofs generalize directly when forcing with the LHMC iteration of the specific fragment of each axiom, but we exemplify the use of "unified" LHMC iterations.

Theorem 5.4. *If $\lambda \geq \kappa$ and κ is λ-weakly compact, then the adding collapsing LHMC iteration of* AAFA *forces* AAFA(λ) *with $\kappa = \mathfrak{c} = \aleph_2$ and λ is not collapsed.*

Proof remarks. Apply the Baumgartner argument, i.e., proceed as in Theorem 4.42 and verify that we can choose our supposed minimal counterexample $\mathbb{Q}$ from the lottery sum in $N[G]$. By the same arguments as there, we know that $\mathbb{Q}$ is a counterexample to AAFA in $N[G]$. As for minimality, if there would be a counterexample to AAFA(μ) $\mathbb{Q}'$ for some $\mu < \lambda$, $\mathbb{Q}'$ would be such a counterexample in $V[G]$ and hence $\mathbb{Q}$ would not have been minimal in the first place. □

Theorem 5.5. *If $\lambda \geq \kappa$ and κ is strongly unfoldable on λ, then the bounded LHMC iteration of* PFA *(Definition 5.1) forces* PFA_λ *where λ is not collapsed.*

Proof remarks. Apply the bounded argument. Set up as in Theorem 4.36. Again we need to choose the supposed minimal counterexample $\mathbb{Q}$ from the lottery sum in $N[G]$. By the usual arguments, $\mathbb{Q}$ is indeed a counterexample to PFA_λ in $N[G]$ and we are concerned with minimality.

Assume $\mathbb{Q}$ is not part of the lottery sum. Let ν be smallest such that PFA_ν fails in $N[G]$ and μ be the minimal hereditary size of a counterexample to PFA_ν in $N[G]$. Then $|\text{TC}(\mathbb{Q})| > \mu$ (otherwise, $\mathbb{Q}$ would be in the lottery sum). Also, because PFA_λ fails as witnessed by $\mathbb{Q}$, $\nu \leq \lambda$. Let $\mathbb{Q}'$ be a counterexample to PFA_ν of hereditary size μ in $N[G]$.

Because $N[G]$ correctly computes properness of forcings hereditarily smaller $\mathbb{Q}$, $\mathbb{Q}'$ is actually (i.e., in $V[G]$) proper, so it is a counterexample to PFA_ν, in particular to PFA_λ, contradicting the minimality of $\mathbb{Q}$. □

One might think now that the result of Section 4.9 would also generalize further. We do not see how we could find a generalization in the same vein as the two theorems above. However, similar to our extension of unfoldables to unfoldables on cardinals, [NS08, Definition 7] generalizes Σ^2_1-indescribables to (ϑ, Σ^2_1)-subcompact cardinals that can be thought of as Σ^2_1-indescribable *intervals*. Then [NS08, Theorem 13] uses (ϑ, Σ^2_1)-subcompactness to generalize the proof and result of Theorem 4.46 to ϑ-linked forcings.

5.3 Open questions and future prospects

We did not find any truly new results. However, it seems reasonable that the LHMC iterations could be used to find so far undiscovered consistency proofs. Most notably, one does not have to deal with finding a fast function, which should open up more large cardinals for investigation. In practice we would suggest to just pretend that a fast function is available and see which axiom the proper lottery preparation produces. Instead of finding the fast function we would then modify the proof to use the LHMC iteration of the axiom we found.

LHMC iterations seem to also be suited to finding new *hierarchies* of forcing axioms and corresponding large cardinal notions. It may be advantageous that one would not have to verify the existence of a fast function for every cardinal in a proposed class of large cardinals. We made some tentative steps towards such results in the previous section. For example, it is known [HJ09] that strongly unfoldable cardinals carry an appropriate fast function to apply a proper lottery preparation, but one would have to investigate this anew for strongly unfoldables on larger cardinals.

The more specific problems we attempted to resolve, but were ultimately unable to do so due to constraints on time, ability or simply the scope of this work, include:

- Can PFA^- and/or $\mathrm{MA}_{\aleph_1}$ be forced via a LHMC iteration without an inaccessible (Sections 4.5, 4.6)?

- What is the consistency strength of the large cardinal notions we defined in Section 5.2?

- Can we improve the techniques of special counterexamples to find directed instead of centered sets (Section 3.3)?

- Can we force PFA($\mathfrak{c}$) with only a weakly compact cardinal (Section 4.9, a question also asked in [NS08])?
- Does anything interesting happen if a LHMC iteration "stops" (Section 3.1)?
- Can we make (under certain assumptions) LHMC iterations more similar to universal iterations in the sense that $\mathbb{P}_\kappa \cap H_\lambda = \mathbb{P}_\kappa \restriction \lambda = \mathbb{P}_\lambda$ (where $\mathbb{P}_\lambda$ is the LHMC iteration of length λ) (Section 4.9)?

More generally speaking, one may also ask if there are any further properties of LHMC iterations themselves not developed here. Also, whether we can draw some advantage from the control of the size of $\mathfrak{c}$ (Section 3.2) is a question that has eluded us so far.

Within the realm of further open questions is, naturally, the usefulness of LHMC iterations beyond the examples presented in this thesis. But we believe that the applicability of these iterations has been sufficiently demonstrated.

6 Bibliography

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