ORDINAL ARITHMETIC

JULIAN J. SCHLÖDER

Abstract. These are verbose tutorial notes on the fundamentals of ordinal arithmetic. We define ordinal arithmetic and give proofs for laws of Left-Monotonicity, Associativity, Distributivity, some minor related properties and the Cantor Normal Form.

1. Ordinals

Definition 1.1. *A set x is called* transitive *iff $\forall y \in x \forall z \in y : z \in x$.*

Definition 1.2. *A set α is called an* ordinal *iff α transitive and all $\beta \in \alpha$ are transitive. Write $\alpha \in \mathrm{Ord}$.*

Lemma 1.3. *If α is an ordinal and $\beta \in \alpha$, then β is an ordinal.*

Proof. β is transitive, since it is in α. Let $\gamma \in \beta$. By transitivity of α, $\gamma \in \alpha$. Hence γ is transitive. Thus β is an ordinal. □

Definition 1.4. *If a is a set, define $a + 1 = a \cup \{a\}$.*

Remark 1.5. *$\emptyset$ is an ordinal. Write $0 = \emptyset$. If α is an ordinal, so is $\alpha + 1$.*

Definition 1.6. *If α and β are ordinals, say $\alpha < \beta$ iff $\alpha \in \beta$.*

Lemma 1.7. *For all ordinals α, $\alpha < \alpha + 1$.*

Proof. $\alpha \in \{\alpha\}$, so $\alpha \in \alpha \cup \{\alpha\} = \alpha + 1$. □

Notation 1.8. *From now on, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ denote ordinals.*

Theorem 1.9. *The ordinals are linearly ordered i.e.*

i. $\forall \alpha : \alpha \not< \alpha$ (strictness).
ii. $\forall \alpha \forall \beta \forall \gamma : \alpha < \beta \land \beta < \gamma \to \alpha < \gamma$ (transitivity).
iii. $\forall \alpha \forall \beta : \alpha < \beta \lor \beta < \alpha \lor \alpha = \beta$ (linearity).

Proof. "i." follows from (Found).

"ii." follows from transitivity of the ordinals.

"iii.": Assume this fails. By (Found), choose a minimal α such that some β is neither smaller, larger or equal to α. Choose the minimal such β. Show towards a contradiction that $\alpha = \beta$:

Let $\gamma \in \alpha$. By minimality of α, $\gamma < \beta \vee \beta < \gamma \vee \beta = \gamma$. If $\beta = \gamma$, $\beta < \alpha \lightning$. If $\beta < \gamma$ then by "ii." $\beta < \alpha \lightning$. Thus $\gamma < \beta$, i.e. $\gamma \in \beta$. Hence $\alpha \subseteq \beta$.

Let $\gamma \in \beta$. By minimality of β, $\gamma < \alpha \vee \alpha < \gamma \vee \alpha = \gamma$. If $\alpha = \gamma$, $\alpha < \beta \lightning$. If $\alpha < \gamma$ then by "ii" $\alpha < \beta \lightning$. Thus $\gamma < \alpha$, i.e. $\gamma \in \alpha$. Hence $\beta \subseteq \alpha$. □

Lemma 1.10. *If $\alpha \neq 0$ is an ordinal, $0 < \alpha$, i.e. 0 is the smallest ordinal.*

Proof. Since $\alpha \neq 0$, by linearity $\alpha < 0$ or $0 < \alpha$, but $\alpha < 0$ would mean $\alpha \in \emptyset \lightning$. □

Definition 1.11. *An ordinal α is called a* successor *iff there is a β with $\alpha = \beta + 1$. Write $\alpha \in \mathrm{Suc}$.*

An ordinal $\alpha \neq \emptyset$ is called a limit *if it is no successor. Write $\alpha \in \mathrm{Lim}$.*

Remark 1.12. *By definition, every ordinal is either $\emptyset$ or a successor or a limit.*

Lemma 1.13. *For all ordinals α, β: If $\beta < \alpha + 1$, then $\beta < \alpha \vee \beta = \alpha$, i.e. $\beta \leq \alpha$.*

Proof. Let $\beta < \alpha + 1$, i.e. $\beta \in \alpha \cup \{\alpha\}$. By definition of $\cup$, $\beta \in \alpha$ or $\beta \in \{\alpha\}$, i.e. $\beta \in \alpha \vee \beta = \alpha$. □

Lemma 1.14. *For all ordinals α, β: If $\beta < \alpha$, then $\beta + 1 \leq \alpha$.*

Proof. Suppose this fails for some α, β. Then by linearity, $\beta + 1 > \alpha$, hence by the previous lemma $\alpha \leq \beta$. Hence by transitivity $\beta < \alpha \leq \beta$, contradicting strictness. □

Lemma 1.15. *For all α, there is no β with $\alpha < \beta < \alpha + 1$.*

Proof. Assume there are such α and β. Then, since $\beta < \alpha + 1$, $\beta \leq \alpha$, but since $\alpha < \beta$, by linearity $\alpha < \alpha$, contradicting strictness. □

Lemma 1.16. *For all α, β, if there is no γ with $\alpha < \gamma < \beta$, then $\beta = \alpha + 1$.*

Proof. Suppose $\beta \neq \alpha + 1$. Since $\alpha < \beta$, $\alpha + 1 \leq \beta$, so $\alpha + 1 < \beta$. Then $\alpha + 1$ is some such γ. □

Lemma 1.17. *The operation $+1 : \mathrm{Ord} \to \mathrm{Ord}$ is injective.*

Proof. Let $\alpha \neq \beta$ be ordinals. Wlog $\alpha < \beta$. Then by the previous lemmas, $\alpha + 1 \leq \beta < \beta + 1$, i.e $\alpha + 1 \neq \beta + 1$. □

Lemma 1.18. $\alpha \in \mathrm{Lim}$ *iff* $\forall \beta < \alpha : \beta + 1 < \alpha$ *and* $\alpha \neq 0$.

Proof. Let $\alpha \in \text{Lim}$, $\beta < \alpha$. By linearity, $\beta + 1 < \alpha \vee \alpha < \beta + 1 \vee \beta + 1 = \alpha$. The last case is excluded by definition of limits. So suppose $\alpha < \beta + 1$. Then $\alpha = \beta \vee \alpha < \beta$.

Since $\beta < \alpha$, $\alpha = \beta$ implies $\alpha < \alpha$, contradicting strictness.

By linearity, $\alpha < \beta$ implies $\beta < \beta$, contradicting strictness.

Thus $\beta + 1 < \alpha$.

Now suppose $\alpha \neq 0$ and $\forall \beta < \alpha : \beta + 1 < \alpha$. Assume $\alpha \in \text{Suc}$. Then there is β such that $\alpha = \beta + 1$. Then $\beta < \beta + 1 = \alpha$, thus $\beta < \alpha$, i.e. $\beta + 1 < \alpha$. Then $\alpha = \beta + 1 < \alpha$, contradicting strictness. Hence α is a limit. $\square$

Theorem 1.19 (Ordinal Induction)**.** *Let φ be a property of ordinals. Suppose the following holds:*

i. $\varphi(\emptyset)$ (base step).
ii. $\forall \alpha : \varphi(\alpha) \to \varphi(\alpha + 1)$ (successor step).
iii. $\forall \alpha \in \text{Lim} : (\forall \beta < \alpha : \varphi(\beta)) \to \varphi(\alpha)$ (limit step).

Then $\varphi(\alpha)$ holds for all ordinals α.

Proof. Suppose i, ii and iii hold. Assume there is some α such that $\neg\varphi(\alpha)$. By (Found), take the smallest such α.

Suppose $\alpha = \emptyset$. This contradicts i.

Suppose $\alpha \in \text{Suc}$. Then there is β such that $\alpha = \beta + 1$, since $\beta < \beta + 1$, $\beta < \alpha$ and hence by minimality of α, $\varphi(\beta)$. By ii, $\varphi(\alpha)$ ↯.

Suppose $\alpha \in \text{Lim}$. By minimality of α, all $\beta < \alpha$ satisfy $\varphi(\beta)$. Thus by iii, $\varphi(\alpha)$ ↯.

Hence there can't be any such α. $\square$

Definition 1.20. *Let ω be the (inclusion-)smallest set that contains* 0 *and is closed under* $+1$*, i.e.* $\forall x \in \omega : x + 1 \in \omega$.

More formally, $\omega = \bigcap\{w \mid 0 \in w \wedge \forall v \in w : v + 1 \in w\}$.

Remark 1.21. *ω is a set by the Axiom of Infinity.*

Theorem 1.22. *ω is an ordinal.*

Proof. Consider $\omega \cap \text{Ord}$. This set contains 0 and is closed under $+1$, as ordinals are closed under $+1$. So ω must by definition be a subset of $\omega \cap \text{Ord}$, i.e. ω contains only ordinals.

Hence it suffices to show that ω is transitive. Consider $\omega' = \{x \mid x \in \omega \wedge \forall y \in x : y \in \omega\}$. Clearly, $0 \in \omega'$. Let $x \in \omega'$ and show that $x + 1 \in \omega'$.

By definition, $x + 1 \in \omega$. Let $y \in x + 1$, i.e. $y = x \vee y \in x$. If $y = x$, y $\in \omega$. If $y \in x$ then $y \in \omega$ by definition of ω'. Hence $x + 1 \in \omega'$.

Thus ω' contains 0 and is closed under $+1$, i.e. $\omega \subseteq \omega'$. But $\omega' \subseteq \omega$ by defintion, hence $\omega = \omega'$, i.e. ω is transitive. $\square$

Theorem 1.23. *ω is a limit, in particular, it is the smallest limit ordinal.*

Proof. $\omega \neq 0$, since $0 \in \omega$. Let $\alpha < \omega$. Then $\alpha + 1 < \omega$ by definition.

Assume $\gamma < \omega$ is a limit ordinal. Since $\gamma \neq \emptyset$, $0 \in \gamma$. Also, as a limit, γ is closed under $+1$. Hence γ contradicts the minimality of ω. □

2. Ordinal Arithmetic

Definition 2.1. *Define an ordinal* $1 := 0 + 1 = \{\emptyset\}$.

Lemma 2.2. $1 \in \omega$.

Proof. $0 \in \omega$ and ω is closed under $+1$. □

Definition 2.3. *Let α, β be ordinals. Define* ordinal addition *recursively:*

i. $\alpha + 0 = \alpha$.
ii. *If* $\beta \in \text{Suc}$, $\beta = \gamma + 1$, *define* $\alpha + \beta = (\alpha + \gamma) + 1$.
iii. *If* $\beta \in \text{Lim}$, *define* $\alpha + \beta = \bigcup_{\gamma < \beta}(\alpha + \gamma)$.

Remark 2.4. *By this definition, the sum $\alpha + 1$ of an ordinal α and the ordinal $1 = \{0\}$ is the same as $\alpha + 1 = \alpha \cup \{\alpha\}$.*

Definition 2.5. *Let α, β be ordinals. Define* ordinal multiplication *recursively:*

i. $\alpha \cdot 0 = 0$.
ii. *If* $\beta \in \text{Suc}$, $\beta = \gamma + 1$, *define* $\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha$.
iii. *If* $\beta \in \text{Lim}$, *define* $\alpha \cdot \beta = \bigcup_{\gamma < \beta}(\alpha \cdot \gamma)$.

Definition 2.6. *Let α, β be ordinals. Define* ordinal exponentiation *recursively:*

i. $\alpha^0 = 1$.
ii. *If* $\beta \in \text{Suc}$, $\beta = \gamma + 1$, *define* $\alpha^\beta = (\alpha^\gamma) \cdot \alpha$.
iii. *If* $\beta \in \text{Lim}$ *and* $\alpha > 0$, *define* $\alpha^\beta = \bigcup_{\gamma < \beta}(\alpha^\gamma)$. *If* $\alpha = 0$, *define* $\alpha^\beta = 0$.

Lemma 2.7. *If A is a set of ordinals, $\bigcup A$ is an ordinal.*

Proof. Let A be a set of ordinals, define $a = \bigcup A$.

Let $x \in y \in a$, then there is an $\alpha \in A$ such that $x \in y \in \alpha$, so $x \in \alpha$ hence $x \in a$. Thus, a is transitive. Let $z \in a$. There is $\alpha \in A$ such that $z \in \alpha$, hence z is transitive.

Thus a is transitive and every element of a is transitive, i.e. a is an ordinal. □

Remark 2.8. *By induction and this lemma, the definitions of* $+$, $\cdot$ *and exponentiation above are well-defined, i.e. if* α, β *are ordinals,* $\alpha+\beta$, $\alpha\cdot\beta$ *and* α^β *are again ordinals.*

Definition 2.9. *Let* A *be a set of ordinals. The* supremum *of* A *is definied as:* $\sup A = \min\{\alpha \mid \forall\beta \in A : \beta \leq \alpha\}$.

Lemma 2.10. *Let* A *be a set of ordinals, then* $\sup A = \bigcup A$.

Proof. Since $\sup A$ is again an ordinal, it is just the set of all ordinals smaller than it. Hence by linearity, $\sup A = \{\alpha \mid \exists\beta \in A : \alpha < \beta\}$. Which equals $\bigcup A$ by definition. □

Lemma 2.11. *Let* A *be a set of ordinals. If* $\sup A$ *is a successor, then* $\sup A \in A$.

Proof. Assume $\sup A = \alpha + 1 \notin A$, then for all $\beta \in A$, $\beta < \alpha + 1$, i.e. $\beta \leq \alpha$. Then $\sup A = \alpha < \alpha + 1 = \sup A$ ↯. □

Lemma 2.12. *Let* A *be a set of ordinals,* $B \subseteq A$ *such that* $\forall\alpha \in A \exists\beta \in B : \alpha \leq \beta$. *Then* $\sup A = \sup B$.

Proof. Show $\{\gamma \mid \forall\alpha \in A : \gamma \geq \alpha\} = \{\gamma \mid \forall\beta \in B : \gamma \geq \beta\}$. Then the minima of these sets, and hence the suprema of A and B, are equal. Suppose $\gamma \geq \alpha$ for all $\alpha \in A$. Then, since $B \subseteq A$, $\gamma \geq \alpha$ for all $\beta \in B$. Suppose $\gamma \geq \beta$ for all $\beta \in B$. Let $\alpha \in A$, then there is some $\beta \in B$ with $\beta \geq \alpha$, hence $\gamma \geq \beta \geq \alpha$. Thus $\gamma \geq \alpha$ for all $\alpha \in A$. □

Lemma 2.13. *If* γ *is a limit,* $\bigcup\gamma = \sup\gamma = \bigcup_{\alpha<\gamma}\alpha = \sup_{\alpha<\gamma}\alpha = \gamma$.

Proof. We've shown a more general form of the first equality, the second and third are just a different ways of writing the same set. Assume $\gamma \neq \sup_{\alpha<\gamma}\alpha$, i.e. $\gamma < \sup_{\alpha<\gamma}\alpha$ or $\sup_{\alpha<\gamma}\alpha < \gamma$ by linearity.

In the first case, there is $\alpha < \gamma$ such that $\gamma < \alpha$, i.e. $\gamma < \gamma$ contradicting strictness.

In the second case, $(\sup_{\alpha<\gamma}\alpha) + 1 < \gamma$, since γ is a limit. But then, by definition of sup, $(\sup_{\alpha<\gamma}\alpha) + 1 \leq \sup_{\alpha<\gamma}\alpha$ while $\sup_{\alpha<\gamma}\alpha < (\sup_{\alpha<\gamma}\alpha) + 1$, again contradicting strictness. □

Lemma 2.14. *For all* α, $0 + \alpha = \alpha$.

Proof. By induction on α. Since $0 + 0 = 0$, the base step is trivial.

Suppose $\alpha = \beta + 1$ and $0 + \beta = \beta$. Then $0 + \alpha = 0 + (\beta + 1) = (0 + \beta) + 1 = \beta + 1 = \alpha$.

Suppose $\alpha \in \mathrm{Lim}$ and for all $\beta < \alpha$, $0 + \beta = \beta$. Then $0 + \alpha = \bigcup_{\beta<\alpha}(0 + \beta) = \bigcup_{\beta<\alpha}\beta = \alpha$. □

Lemma 2.15. *For all* α, $1 \cdot \alpha = \alpha \cdot 1 = \alpha$.

Proof. $\alpha \cdot 1 = \alpha \cdot (0+1) = (\alpha \cdot 0) + \alpha = \alpha$. Prove $1 \cdot \alpha = \alpha$ by induction on α. Since $1 \cdot 0 = 0$, the base step holds.

Suppose $\alpha = \beta + 1$ and $1 \cdot \beta = \beta$. Then $1 \cdot \alpha = (1 \cdot \beta) + 1 = \beta + 1 = \alpha$.

Suppose α is a limit and for all $\beta < \alpha$, $1 \cdot \beta = \beta$. Then $1 \cdot \alpha = \bigcup_{\beta<\alpha}(1 \cdot \beta) = \bigcup_{\beta<\alpha} \beta = \alpha$. □

Lemma 2.16. *For all α, $\alpha^1 = \alpha$.*

Proof. $\alpha^1 = \alpha^{0+1} = \alpha^0 \cdot \alpha = 1 \cdot \alpha = \alpha$. □

Lemma 2.17. *For all α, $1^\alpha = 1$.*

Proof. If $\alpha = 0$, $1^\alpha = 1$ by definition. If $\alpha = \beta + 1$, $1^{\beta+1} = 1^\beta \cdot 1 = 1$.

If α is a limit, $1^\alpha = \sup_{\beta<\alpha} 1^\beta = \sup_{\beta<\alpha} 1 = 1$. □

Lemma 2.18. *Let α be an ordinal. If $\alpha > 0$, $0^\alpha = 0$. Otherwise $0^\alpha = 1$.*

Proof. $0^0 = 1$ by definition, so let $\alpha > 0$. If $\alpha = \beta + 1$, $0^\alpha = 0^\beta \cdot 0 = 0$. If α is a limit, $0^\alpha = 0$ by defnition. □

Theorem 2.19 (Subtraction)**.** *For all $\beta \leq \alpha$ there is some $\gamma \leq \alpha$ with $\beta + \gamma = \alpha$.*

Proof. By induction on α. $\alpha = 0$ is trivial. Suppose $\alpha = \delta + 1$ and $\beta \leq \alpha$. If $\beta = \alpha$, set $\gamma = 0$. So suppose $\beta < \alpha$, i.e. $\beta \leq \delta$. Find $\gamma' \leq \beta$ with $\beta + \gamma' = \delta$. Set $\gamma = \gamma' + 1$, then $\beta + \gamma = \beta + (\gamma' + 1) = (\beta + \gamma') + 1 = \delta + 1 = \alpha$.

If α is a limit and $\beta < \alpha$ then for all $\delta < \alpha$, $\beta \leq \delta$, find γ_δ such that $\beta + \gamma_\delta = \delta$. If $\delta < \beta$, set $\gamma_\delta = 0$. Set $\gamma = \sup_{\beta<\delta<\alpha} \gamma_\delta$. If γ is a successor, then there is some δ with $\gamma = \gamma_\delta$. But $\delta + 1 < \alpha$ and as in the successor case, $\gamma_{\delta+1} = \gamma_\delta + 1 > \gamma_\delta = \gamma$, so this can't be the supremum.

Also, $\gamma \neq 0$, since if it were, for all $\beta < \delta < \gamma$, $\beta = \delta$, i.e. there are no such δ. This implies $\beta + 1 = \alpha$, but α is no successor.

So, γ is a limit. In particular for all $\delta < \alpha$, $\gamma_\delta < \gamma$: If there were any $\delta < \gamma$ with $\gamma_\delta = \gamma$, then since $\gamma \neq 0$, $\beta < \delta$. Then again $\gamma_{\delta+1} = \gamma_\delta + 1 > \gamma_\delta = \gamma$, contradicting that γ is the supremum. Hence, $\beta + \gamma = \sup_{\varepsilon<\gamma}(\beta + \varepsilon) = \sup_{\gamma_\delta<\gamma}(\beta + \gamma_\delta) = \sup_{\gamma_\delta<\gamma} \delta = \sup_{\delta<\alpha} \delta = \alpha$. □

Theorem 2.20. *ω is closed under $+$, $\cdot$ and exponentiation, i.e. $\forall n, m \in \omega : n + m \in \omega \wedge n \cdot m \in \omega \wedge n^m \in \omega$.*

Proof. By induction on m. Since ω does not contain any limits, we may omit the limit step.

First consider addition. If $m = 0$ then $n + m = m \in \omega$. Suppose $m = k + 1$. $n + m = (n + k) + 1$. By induction $n + k \in \omega$ and since ω is closed under $+1$, $(n + k) + 1 \in \omega$.

Now consider multiplication. If $m = 0$, $n \cdot 0 = 0 \in \omega$. Suppose $m = k + 1$. $n \cdot m = (n \cdot k) + n$. By induction $n \cdot k \in \omega$ and since ω is closed under $+$, $(n \cdot k) + n \in \omega$.

Finally consider exponentiation. If $m = 0$, $n^0 = 1 \in \omega$. Suppose $m = k + 1$. $n^m = n^k \cdot n$. By induction, $n^k \in \omega$ and since ω is closed under $\cdot$, $n^k \cdot n \in \omega$. $\square$

3. Monotonicity Laws

3.1. Comparisons of Addition.

Lemma 3.1. *If α and β are ordinals, and $\alpha \leq \beta$, then $\alpha + 1 \leq \beta + 1$.*

Proof. Assume $\alpha \leq \beta$ and $\alpha + 1 > \beta + 1$. By transitivity it suffices to now derive a contradiction. Since $\beta + 1 < \alpha + 1$, $\beta + 1 = \alpha \vee \beta + 1 < \alpha$.

If $\beta + 1 = \alpha$, $\beta + 1 \leq \beta$, but $\beta < \beta + 1$ ↯.

If $\beta + 1 < \alpha$, by transitivity $\beta + 1 \leq \beta$, but $\beta < \beta + 1$ ↯. $\square$

Lemma 3.2. *If α and β are ordinals, then $\alpha \leq \alpha + \beta$.*

Proof. By induction on β: If $\beta = 0$, $\alpha = \alpha + \beta$.

If $\beta = \gamma + 1$ and $\alpha \leq \alpha + \gamma$, then $\alpha + \beta = (\alpha + \gamma) + 1 \geq \alpha + 1 \geq \alpha$.

If $\beta \in \text{Lim}$ and for all $\gamma < \beta$, $\alpha \leq \alpha + \gamma$, then: $\alpha + \beta = \bigcup_{\gamma<\beta}(\alpha + \gamma) = \sup_{\gamma<\beta}(\alpha + \gamma) \geq \sup_{\gamma<\beta} \alpha = \alpha$. $\square$

Lemma 3.3. *If α and β are ordinals, then $\beta \leq \alpha + \beta$.*

Proof. By induction on β: $\beta = 0$ is trivial, since 0 is the smallest ordinal.

If $\beta = \gamma + 1$ and $\gamma \leq \alpha + \gamma$, then $\alpha + \beta = (\alpha + \gamma) + 1 \geq \gamma + 1 = \beta$.

If $\beta \in \text{Lim}$ and for all $\gamma < \beta$, $\gamma \leq \alpha + \gamma$, then: $\alpha + \beta = \bigcup_{\gamma<\beta}(\alpha + \gamma) = \sup_{\gamma<\beta}(\alpha + \gamma) \geq \sup_{\gamma<\beta} \gamma = \beta$. $\square$

Lemma 3.4. *If γ is a limit, then for all α: $\alpha + \gamma$ is a limit.*

Proof. $\gamma \neq 0$, so $\alpha + \gamma \geq \gamma > 0$, i.e. $\alpha + \gamma \neq 0$. So let $x \in \alpha + \gamma$. Show that $x + 1 < \alpha + \gamma$.

$x \in \alpha + \gamma = \bigcup_{\beta<\gamma}(\alpha + \beta)$, i.e. there is $\beta < \gamma$ such that $x \in \alpha + \beta$. By a previous lemma, $x + 1 \leq \alpha + \beta$. If $x + 1 \in \alpha + \beta$, $x + 1 < \alpha + \gamma$.

So suppose $\alpha + \beta = x + 1$. Since γ is a limit, $\beta + 1 < \gamma$ and by definition $\alpha + (\beta + 1) = (\alpha + \beta) + 1$, and $x + 1 \in (\alpha + \beta) + 1$, hence $x + 1 \in \alpha + \gamma$. $\square$

Lemma 3.5. *Suppose γ is a limit, α, β are ordinals and $\beta < \gamma$. then $\alpha + \beta < \alpha + \gamma$.*

Proof. By definition, $\alpha+\gamma = \bigcup_{\delta<\gamma}(\alpha+\delta)$. Since γ is a limit, $\beta+1 < \gamma$. Also by definition: $\alpha+\beta < (\alpha+\beta)+1 = \alpha+(\beta+1) \in \{\alpha+\delta \mid \delta < \gamma\}$. Hence $\alpha + \beta \in \bigcup_{\delta<\gamma} \alpha + \delta$. □

Lemma 3.6. *Suppose α, β, γ are ordinals and $\beta < \gamma$. then $\alpha + \beta < \alpha + \gamma$.*

Proof. By induction over γ. $\gamma = 0$ is clear, since there is no $\beta < 0$. And the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta+1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, notice: $\alpha + \beta = \alpha + \delta < (\alpha + \delta) + 1 = \alpha + (\delta + 1) = \alpha + \gamma$.

If $\beta < \delta$, apply induction: $\alpha+\beta < \alpha+\delta$. Hence $\alpha+\beta < (\alpha+\delta)+1 = \alpha + (\delta + 1) = \alpha + \gamma$. □

Theorem 3.7 (Left-Monotonicity of Ordinal Addition)**.** *Let α, β, γ be ordinals. The following are equivalent:*

i. $\beta < \gamma$.
ii. $\alpha + \beta < \alpha + \gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha + \beta < \alpha + \gamma$ and not $\beta < \gamma$. By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha + \gamma = \alpha + \beta < \alpha + \gamma$↯. If $\gamma < \beta$, by the forward direction, $\alpha + \gamma < \alpha + \beta < \alpha + \gamma$↯. □

Lemma 3.8. $1 + \omega = \omega$.

Proof. $1 + \omega$ is a limit by a lemma above, so $\omega \leq 1 + \omega$ (since ω is the smallest limit). ω is a limit, so $1 + \omega = \sup_{\alpha<\omega} 1 + \alpha$. Since ω is closed under $+$, $1 + \alpha < \omega$ for all $\alpha < \omega$, hence $\sup_{\alpha<\omega} \leq \omega$. It follows that $1 + \omega = \omega$. □

Remark 3.9. *Right-Monotoniticy does not hold: Clearly, $0 < 1$ and we've seen that $0 + \omega = \omega$ and $1 + \omega = \omega$. So $0 + \omega \not< 1 + \omega$.*

3.2. **Comparisons of Multiplications.**

Lemma 3.10. *For all α, β: $\alpha + \beta = 0$ iff $\alpha = \beta = 0$.*

Proof. Reverse direction is trivial. So suppose $\alpha + \beta = 0$ and not $\alpha = \beta = 0$. If $\beta = 0$, then $0 = \alpha + \beta = \alpha$ and if $\beta > 0$, by Left-Monotonicity $0 \leq \alpha + 0 < \alpha + \beta = 0$↯. □

Lemma 3.11. *If α and $\beta \neq 0$ are ordinals, then $\alpha \leq \alpha \cdot \beta$.*

Proof. By induction on β. Suppose $\beta = \gamma+1$, then $\alpha\cdot\beta = (\alpha\cdot\gamma)+\alpha \geq \alpha$, by induction and the corresponding lemma on addition.

Suppose β is a limit. $\alpha\cdot\beta = \bigcup_{\gamma<\beta}(\alpha\cdot\gamma) = \sup_{\gamma<\beta}(\alpha\cdot\gamma) \geq \sup_{\gamma<\beta} \alpha = \alpha$. □

Lemma 3.12. *If $\alpha \neq 0$ and β are ordinals, then $\beta \leq \alpha \cdot \beta$.*

Proof. By induction on β. $\beta = 0$ is trivial. Suppose $\beta = \gamma + 1$, then:

$$\begin{aligned} \alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha > \alpha \cdot \gamma \quad & \text{(since } \alpha > 0 \text{ and by Left-Monotonicity)} \\ \geq \gamma \quad & \text{(by induction).} \end{aligned}$$

And $\alpha \cdot \beta > \gamma$ implies $\alpha \cdot \beta \geq \gamma + 1 = \beta$.

Suppose β is a limit. $\alpha \cdot \beta = \bigcup_{\gamma<\beta}(\alpha \cdot \gamma) = \sup_{\gamma<\beta}(\alpha \cdot \gamma) \geq \sup_{\gamma<\beta} \gamma = \beta$. □

Lemma 3.13. *If γ is a limit, then for all $\alpha \neq 0$: $\alpha \cdot \gamma$ is a limit.*

Proof. $\gamma \neq 0$, so $\alpha \cdot \gamma \geq \gamma > 0$, i.e. $\alpha \cdot \gamma \neq 0$. So let $x \in \alpha \cdot \gamma$. Show that $x + 1 < \alpha \cdot \gamma$.

$x \in \alpha \cdot \gamma = \bigcup_{\beta<\gamma}(\alpha \cdot \beta)$, i.e. there is $\beta < \gamma$ such that $x \in \alpha \cdot \beta$. By a previous lemma, $x + 1 \leq \alpha \cdot \beta$. If $x + 1 \in \alpha \cdot \beta$, $x + 1 < \alpha \cdot \gamma$.

So suppose $\alpha \cdot \beta = x + 1$. Since γ is a limit, $\beta + 1 < \gamma$ and by definition $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, and $x + 1 \in (\alpha \cdot \beta) + 1 \leq (\alpha \cdot \beta) + \alpha$ by Left-Monotonicity (since $\alpha \geq 1$). Hence $x + 1 \in \alpha \cdot \gamma$. □

Lemma 3.14. *Suppose γ is a limit, $\alpha \neq 0$ and β are ordinals and $\beta < \gamma$. Then $\alpha \cdot \beta < \alpha \cdot \gamma$.*

Proof. By definition, $\alpha \cdot \gamma = \bigcup_{\delta<\gamma}(\alpha \cdot \delta)$. Since γ is a limit, $\beta + 1 < \gamma$. By Left-Monotonicity: $\alpha \cdot \beta < (\alpha \cdot \beta) + \alpha = \alpha \cdot (\beta + 1) \in \{\alpha \cdot \delta \mid \delta < \gamma\}$. Hence $\alpha \cdot \beta \in \bigcup_{\delta<\gamma} \alpha \cdot \delta$. □

Lemma 3.15. *Suppose $\alpha \neq 0$ and β, γ are ordinals and $\beta < \gamma$. Then $\alpha \cdot \beta < \alpha \cdot \gamma$.*

Proof. By induction over γ. $\gamma = 0$ is clear and the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta + 1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, apply Left-Monotonicity: $\alpha \cdot \beta = \alpha \cdot \delta < (\alpha \cdot \delta) + \alpha = \alpha \cdot (\delta + 1) = \alpha \cdot \gamma$.

If $\beta < \delta$, apply induction: $\alpha \cdot \beta < \alpha \cdot \delta$. Hence (by Left-Monotonicity) $\alpha \cdot \beta < \alpha \cdot \delta < (\alpha \cdot \delta) + \alpha = \alpha \cdot (\delta + 1) = \alpha \cdot \gamma$. □

Theorem 3.16 (Left-Monotonicity of Ordinal Multiplication)**.** *Let α, β, γ be ordinals. The following are equivalent:*

i. $\beta < \gamma \wedge \alpha > 0$.
ii. $\alpha \cdot \beta < \alpha \cdot \gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha \cdot \beta < \alpha \cdot \gamma$ and not $\beta < \gamma$. If $\alpha = 0$, then $\alpha \cdot \beta = 0 = \alpha \cdot \gamma \lightning$. So $\alpha > 0$. By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha \cdot \gamma = \alpha \cdot \beta < \alpha \cdot \gamma \lightning$. If $\gamma < \beta$, by the forward direction, $\alpha \cdot \gamma < \alpha \cdot \beta < \alpha \cdot \gamma \lightning$. □

Lemma 3.17. *Let* $2 = 1 + 1$. $2 \cdot \omega = \omega$.

Proof. Since ω is the smallest limit and by a lemma above $2 \cdot \omega$ is a limit, $\omega \leq 2 \cdot \omega$. Since ω is closed under $\cdot$, for all $\alpha < \omega$, $2 \cdot \alpha \in \omega$. Hence $2 \cdot \omega = \sup_{\alpha<\omega} 2 \cdot \alpha \leq \omega$. □

Remark 3.18. *Right-Monotonicity does not hold: Clearly,* $1 < 2$ *and since* $1 \cdot \omega = \omega$ *and* $2 \cdot \omega = \omega$, $1 \cdot \omega \not< 2 \cdot \omega$.

3.3. Comparisons of Exponentation.

Lemma 3.19. *If* α *and* $\beta \neq 0$ *are ordinals, then* $\alpha \leq \alpha^\beta$.

Proof. If $\alpha = 0$ the lemma is trivial. So suppose $\alpha > 0$.

By induction on β. Suppose $\beta = \gamma + 1$, then $\alpha^\beta = (\alpha^\gamma) \cdot \alpha \geq \alpha$, by induction and the corresponding lemma on multiplication.

Suppose β is a limit. $\alpha^\beta = \bigcup_{\gamma<\beta}(\alpha^\gamma) = \sup_{\gamma<\beta}(\alpha^\gamma) \geq \sup_{\gamma<\beta} \alpha = \alpha$. □

Lemma 3.20. *If* $\alpha > 1$ *and* β *are ordinals, then* $\beta \leq \alpha^\beta$.

Proof. If $\beta = 0$ the lemma is trivial. So suppose $\beta > 0$.

By induction on β. Suppose $\beta = \gamma + 1$, then:

$$\begin{aligned} \alpha^\beta = (\alpha^\gamma) \cdot \alpha &> \alpha^\gamma \cdot 1 && \text{(by Left-Monotonicity)} \\ &= \alpha^\gamma \geq \gamma && \text{(by induction).} \end{aligned}$$

And $\alpha^\beta > \gamma$ implies $\alpha^\beta \geq \gamma + 1 = \beta$.

Suppose β is a limit. $\alpha^\beta = \bigcup_{\gamma<\beta}(\alpha^\gamma) = \sup_{\gamma<\beta}(\alpha^\gamma) \geq \sup_{\gamma<\beta} \gamma = \beta$. □

Lemma 3.21. *If* γ *is a limit, then for all* $\alpha > 1$*:* α^γ *is a limit.*

Proof. $\gamma \neq 0$, so $\alpha^\gamma \geq \gamma > 0$, i.e. $\alpha^\gamma \neq 0$. So let $x \in \alpha^\gamma$. Show that $x + 1 < \alpha^\gamma$.

$x \in \alpha^\gamma = \bigcup_{\beta<\gamma}(\alpha^\beta)$, i.e. there is $\beta < \gamma$ such that $x \in \alpha^\beta$. By a previous lemma, $x + 1 \leq \alpha^\beta$. If $x + 1 \in \alpha^\beta$, $x + 1 < \alpha^\gamma$.

So suppose $\alpha^\beta = x+1$. Since γ is a limit, $\beta+1 < \gamma$ and by definition $\alpha^{\beta+1} = (\alpha^\beta) \cdot \alpha$, and $x + 1 \in (\alpha^\beta) + 1 \leq \alpha^\beta + \alpha^\beta \leq \alpha^\beta \cdot 2 \leq \alpha^\beta \cdot \alpha$ by Left-Monotonicity (since $\alpha \geq 2$). Hence $x + 1 \in \alpha \cdot \gamma$. □

Lemma 3.22. *Suppose* γ *is a limit,* $\alpha > 1$ *and* β *are ordinals and* $\beta < \gamma$. *Then* $\alpha^\beta < \alpha^\gamma$.

Proof. By definition, $\alpha^\gamma = \bigcup_{\delta<\gamma}(\alpha^\delta)$. Since γ is a limit, $\beta + 1 < \gamma$. By Left-Monotonicity: $\alpha^\beta < (\alpha^\beta) \cdot \alpha = \alpha^{\beta+1} \in \{\alpha^\delta \mid \delta < \gamma\}$. Hence $\alpha^\beta \in \bigcup_{\delta<\gamma} \alpha^\delta$. □

Lemma 3.23. *Suppose $\alpha > 1$ and β, γ are ordinals and $\beta < \gamma$. Then $\alpha^\beta < \alpha^\gamma$.*

Proof. By induction over γ. $\gamma = 0$ is clear and the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta+1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, apply Left-Monotonicity: $\alpha^\beta = \alpha^\delta < (\alpha^\delta) \cdot \alpha = \alpha^{\delta+1} = \alpha^\gamma$.

If $\beta < \delta$, apply induction: $\alpha^\beta < \alpha^\delta$. Hence (by Left-Monotonicity) $\alpha^\beta < \alpha^\delta < (\alpha^\delta) \cdot \alpha = \alpha^{\delta+1} = \alpha^\gamma$. □

Theorem 3.24 (Left-Monotonicity of Ordinal Exponentiation)**.** *Let α, β, γ be ordinals and $\alpha > 0$. The following are equivalent:*

i. $\beta < \gamma \wedge \alpha > 1$.
ii. $\alpha^\beta < \alpha^\gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha^\beta < \alpha^\gamma$ and not $\beta < \gamma$. If $\alpha = 1$, $\alpha^\beta = 1 = \alpha^\gamma \lightning$.

By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha^\gamma = \alpha^\beta < \alpha^\gamma \lightning$. If $\gamma < \beta$, by the forward direction, $\alpha^\gamma < \alpha^\beta < \alpha^\gamma \lightning$. □

Lemma 3.25. *Let $0 < n \in \omega$. $n^\omega = \omega$.*

Proof. ω is the smallest limit and n^ω is a limit by a lemma above. So $\omega \leq n^\omega$. Since ω is closed under exponentiation, for all $\alpha < \omega$, $n^\alpha \in \omega$. Then $n^\omega = \sup_{\alpha<\omega} n^\alpha \leq \omega$. □

Remark 3.26. *Right-Monotonicity does not hold: Define $3 = 2 + 1 \in \omega$. Clearly, $2 < 3$ and since $2^\omega = \omega$ and $3^\omega = \omega$, $2^\omega \not< 3^\omega$.*

4. Associativity, Distributivity and Commutativity

Theorem 4.1. *$+, \cdot$ and exponentiation are not commutative, i.e. there are $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ such that $\alpha + \beta \neq \beta + \alpha$, $\gamma \cdot \delta \neq \delta \cdot \gamma$ and $\varepsilon^\zeta \neq \zeta^\varepsilon$.*

Proof. Let $\alpha = 1$, $\beta = \omega$, $\gamma = 2$, $\delta = \omega$, $\varepsilon = 0$, $\zeta = 1$.

$1 + \omega = \omega$ as shown above. $\omega \in \omega \cup \{\omega\} = \omega + 1$, so $\alpha + \beta < \beta + \alpha$.

$2 \cdot \omega = \omega$ as shown above. By Left-Monotonicity, $\omega < \omega + \omega = \omega \cdot 2$. So $\gamma \cdot \delta < \delta \cdot \gamma$.

$0^1 = 0^0 \cdot 0 = 0$, but $1^0 = 1$ by definition. Hence $\varepsilon^\zeta < \zeta^\varepsilon$. □

Theorem 4.2 (Associativity of Ordinal Addition)**.** *Let α, β, γ be ordinals. Then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.*

Proof. By induction on γ. $\gamma = 0$ is trivial. Suppose $\gamma = \delta + 1$.

$$\begin{aligned}(\alpha+\beta)+(\delta+1) &= ((\alpha+\beta)+\delta)+1 && \text{(by definition)}\\ &= (\alpha+(\beta+\delta))+1 && \text{(by induction)}\\ &= \alpha+((\beta+\delta)+1) && \text{(by definition)}\\ &= \alpha+(\beta+(\delta+1)) && \text{(by definition)}\\ &= \alpha+(\beta+\gamma).\end{aligned}$$

Now suppose γ is a limit, in particular $\gamma > 1$. Then $\beta + \gamma$ is a limit, so $\alpha + (\beta + \gamma)$ and $(\alpha + \beta) + \gamma$ are limits.

$$\begin{aligned}(\alpha+\beta)+\gamma &= \sup_{\varepsilon<\gamma}((\alpha+\beta)+\varepsilon) && \text{(by definition)}\\ &= \sup_{\beta+\varepsilon<\beta+\gamma}((\alpha+\beta)+\varepsilon) && \text{(by Left-Monotonicity)}\\ &= \sup_{\beta+\varepsilon<\beta+\gamma}(\alpha+(\beta+\varepsilon)) && \text{(by induction)}\\ &= \sup_{\delta<\beta+\gamma}(\alpha+\delta) && \text{(see below)}\\ &= \alpha+(\beta+\gamma) && \text{(by definition).}\end{aligned}$$

Recall Lemma 2.12. Write $B = \{\alpha + (\beta + \varepsilon) \mid \beta + \varepsilon < \beta + \gamma\}$ and $A = \{\alpha + \delta \mid \delta < \beta + \gamma\}$. Clearly $B \subseteq A$.

Let $\alpha + \delta \in A$. Let $\varepsilon = \min\{\zeta \mid \beta + \zeta \geq \delta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\zeta < \gamma$, $\beta + \zeta < \delta$. Then $\delta < \beta + \gamma = \sup_{\zeta<\gamma} \beta + \zeta \leq \delta ↯$. Hence, $\varepsilon < \gamma$, i.e. $\beta + \varepsilon < \beta + \gamma$. By construction, $\delta \leq \beta + \varepsilon$. Thus, by Left-Monotonicity, $\alpha + \delta \leq \alpha + (\beta + \varepsilon) \in B$. Thus, the conditions of Lemma 2.12 are satisfied. □

Theorem 4.3 (Distributivity)**.** *Let α, β, γ be ordinals. Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.*

Proof. Note that the theorem is trivial if $\alpha = 0$, so suppose $\alpha > 0$. Proof by induction on γ. $\gamma = 0$ is trivial. Suppose $\gamma = \delta + 1$.

$$\begin{aligned}\alpha\cdot(\beta+(\delta+1)) &= \alpha\cdot((\beta+\delta)+1) && \text{(by definition)}\\ &= \alpha\cdot(\beta+\delta)+\alpha && \text{(by definition)}\\ &= \alpha\cdot\beta+\alpha\cdot\delta+\alpha && \text{(by induction)}\\ &= \alpha\cdot\beta+\alpha\cdot(\delta+1) && \text{(by definition).}\end{aligned}$$

Suppose γ is a limit. Hence $\alpha \cdot \gamma$ and $\beta + \gamma$ are limits.

$$\begin{aligned}
\alpha \cdot (\beta + \gamma) &= \sup_{\delta < \beta+\gamma} \alpha \cdot \delta && \text{(by definition)} \\
&= \sup_{\beta+\varepsilon < \beta+\gamma} (\alpha \cdot (\beta + \varepsilon)) && \text{(see below)} \\
&= \sup_{\varepsilon < \gamma} (\alpha \cdot (\beta + \varepsilon)) && \text{(by Left-Monotonicity)} \\
&= \sup_{\varepsilon < \gamma} (\alpha \cdot \beta + \alpha \cdot \varepsilon) && \text{(by induction)} \\
&= \sup_{\alpha\cdot\varepsilon < \alpha\cdot\gamma} (\alpha \cdot \beta + \alpha \cdot \varepsilon) && \text{(by Left-Monotonicity)} \\
&= \sup_{\zeta < \alpha\cdot\gamma} (\alpha \cdot \beta + \zeta) && \text{(see below)} \\
&= \alpha \cdot \beta + \alpha \cdot \gamma && \text{(by definition).}
\end{aligned}$$

Recall Lemma 2.12. Write $B = \{\alpha \cdot (\beta + \varepsilon) \mid \beta + \varepsilon < \beta + \gamma\}$ and $A = \{\alpha \cdot \delta \mid \delta < \beta + \gamma\}$. Clearly $B \subseteq A$. Let $\alpha \cdot \delta \in A$. Let $\varepsilon = \min\{\eta \mid \beta + \eta \geq \delta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\beta + \eta < \delta$. Then $\delta < \beta + \gamma = \sup_{\eta<\gamma} \beta + \eta \leq \delta \lightning$. Hence, $\varepsilon < \gamma$, i.e. $\beta + \varepsilon < \beta + \gamma$. By construction, $\delta \leq \beta + \varepsilon$. Thus, by Left-Monotonicity, $\alpha \cdot \delta \leq \alpha \cdot (\beta + \varepsilon) \in B$. Thus, the conditions of Lemma 2.12 are satisfied.

Write $B = \{\alpha \cdot \beta + \alpha \cdot \varepsilon \mid \alpha \cdot \varepsilon < \alpha \cdot \gamma\}$ and $A = \{\alpha \cdot \beta + \zeta \mid \zeta < \alpha \cdot \gamma\}$. Clearly $B \subseteq A$. Let $\alpha \cdot \beta + \zeta \in A$. Let $\varepsilon = \min\{\eta \mid \alpha \cdot \eta \geq \zeta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\alpha \cdot \eta < \zeta$. Then $\zeta < \alpha \cdot \gamma = \sup_{\eta<\gamma} \alpha \cdot \eta \leq \zeta \lightning$. Hence, $\varepsilon < \gamma$, i.e. $\alpha \cdot \varepsilon < \alpha \cdot \gamma$. By construction, $\zeta \leq \alpha \cdot \varepsilon$. Thus, by Left-Monotonicity, $\alpha \cdot \beta + \zeta \leq \alpha \cdot \beta + \alpha \cdot \varepsilon \in B$. Thus, the conditions of Lemma 2.12 are satisfied. ☐

Theorem 4.4 (Associativity of Ordinal Multiplication)**.** *Let α, β, γ be ordinals. Then $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$.*

Proof. Note that the theorem is trivial if $\beta = 0$. So suppose $\beta > 0$. Proof by induction on γ. $\gamma = 0$ is trivial. Suppose $\gamma = \delta + 1$.

$$\begin{aligned}
(\alpha \cdot \beta) \cdot (\delta + 1) &= ((\alpha \cdot \beta) \cdot \delta) + (\alpha \cdot \beta) && \text{(by definition)} \\
&= (\alpha \cdot (\beta \cdot \delta)) + (\alpha \cdot \beta) && \text{(by induction)} \\
&= \alpha \cdot ((\beta \cdot \delta) + \beta) && \text{(by Distributivity)} \\
&= \alpha \cdot (\beta \cdot (\delta + 1)) && \text{(by definition)} \\
&= \alpha \cdot (\beta \cdot \gamma).
\end{aligned}$$

Now suppose γ is a limit, in particular $\gamma > 1$. Then $\beta \cdot \gamma$ is a limit, so $\alpha \cdot (\beta \cdot \gamma)$ and $(\alpha \cdot \beta) \cdot \gamma$ are limits.

$$\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= \sup_{\varepsilon<\gamma}((\alpha \cdot \beta) \cdot \varepsilon) && \text{(by definition)} \\
&= \sup_{\beta\cdot\varepsilon<\beta\cdot\gamma}((\alpha \cdot \beta) \cdot \varepsilon) && \text{(by Left-Monotonicity)} \\
&= \sup_{\beta\cdot\varepsilon<\beta\cdot\gamma}(\alpha \cdot (\beta \cdot \varepsilon)) && \text{(by induction)} \\
&= \sup_{\delta<\beta\cdot\gamma}(\alpha \cdot \delta) && \text{(see below)} \\
&= \alpha \cdot (\beta \cdot \gamma) && \text{(by definition).}
\end{aligned}$$

Recall Lemma 2.12. Write $B = \{\alpha \cdot (\beta \cdot \varepsilon) \mid \beta \cdot \varepsilon < \beta \cdot \gamma\}$ and $A = \{\alpha \cdot \delta \mid \delta < \beta \cdot \gamma\}$. Clearly $B \subseteq A$. If $A = \emptyset$, $B = A$.

Let $\alpha \cdot \delta \in A$. Let $\varepsilon = \min\{\zeta \mid \beta \cdot \zeta \geq \delta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\zeta < \gamma$, $\beta \cdot \zeta < \delta$. Then $\delta < \beta \cdot \gamma = \sup_{\zeta<\gamma} \beta \cdot \zeta \leq \delta \lightning$. Hence, $\varepsilon < \gamma$, i.e. $\beta \cdot \varepsilon < \beta \cdot \gamma$. By construction, $\delta \leq \beta \cdot \varepsilon$. Thus, by Left-Monotonicity, $\alpha \cdot \delta \leq \alpha \cdot (\beta \cdot \varepsilon) \in B$. Thus, the conditions of Lemma 2.12 are satisfied. ☐

Notation 4.5. *As of now, we may omit bracketing ordinal addition and multiplication.*

Remark 4.6. *Ordinal exponentiation is not associative, i.e. there are* α, β, γ *with* $\alpha^{(\beta^\gamma)} \neq (\alpha^\beta)^\gamma$.

Proof. Let $\alpha = \omega$, $\beta = 1$, $\gamma = \omega$. Then $\beta^\gamma = 1$, i.e. $\alpha^{(\beta^\gamma)} = \alpha^1 = \omega$. But $\alpha^\beta = \omega$, hence $(\alpha^\beta)^\gamma = \omega^\omega$. And $\omega < \omega^\omega$ by Left-Monotonicity. ☐

Theorem 4.7. *Let* α, β, γ *be ordinals. Then* $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.

Proof. Recall that $\beta + \gamma = 0$ iff $\beta = \gamma = 0$, so the theorem holds for $\alpha = 0$. Also note that the theorem is trivial for $\alpha = 1$, so suppose $\alpha > 1$. Proof by induction on γ. $\gamma = 0$ is trivial. Suppose $\gamma = \delta + 1$.

$$\begin{aligned}
\alpha^{\beta+\delta+1} &= \alpha^{\beta+\delta} \cdot \alpha && \text{(by definition)} \\
&= \alpha^\beta \cdot \alpha^\delta \cdot \alpha && \text{(by induction)} \\
&= \alpha^\beta \cdot \alpha^{\delta+1} && \text{(by definition).}
\end{aligned}$$

Suppose γ is a limit. Then α^γ and $\alpha^{\beta+\gamma}$ are limits.

$$\begin{aligned}
\alpha^{\beta+\gamma} &= \sup_{\delta<\beta+\gamma} \alpha^\delta && \text{(by definition)}\\
&= \sup_{\beta+\varepsilon<\beta+\gamma} \alpha^{\beta+\varepsilon} && \text{(see below)}\\
&= \sup_{\varepsilon<\gamma} \alpha^{\beta+\varepsilon} && \text{(by Left-Monotonicity)}\\
&= \sup_{\varepsilon<\gamma}(\alpha^\beta \cdot \alpha^\varepsilon) && \text{(by induction)}\\
&= \sup_{\alpha^\varepsilon<\alpha^\gamma}(\alpha^\beta \cdot \alpha^\varepsilon) && \text{(by Left-Monotonicity)}\\
&= \sup_{\zeta<\alpha^\gamma}(\alpha^\beta \cdot \zeta) && \text{(see below)}\\
&= \alpha^\beta + \alpha^\gamma && \text{(by definition).}
\end{aligned}$$

Recall Lemma 2.12. Write $B = \{\alpha^{\beta+\varepsilon} \mid \beta+\varepsilon < \beta+\gamma\}$ and $A = \{\alpha^\delta \mid \delta < \beta+\gamma\}$. Clearly $B \subseteq A$. Let $\alpha^\delta \in A$. Let $\varepsilon = \min\{\eta \mid \beta+\eta \geq \delta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\beta+\eta < \delta$. Then $\delta < \beta+\gamma = \sup_{\eta<\gamma} \beta+\eta \leq \delta \lightning$. Hence, $\varepsilon < \gamma$, i.e. $\beta+\varepsilon < \beta+\gamma$. By construction, $\delta \leq \beta+\varepsilon$. Thus, by Left-Monotonicity, $\alpha^\delta \leq \alpha^{\beta+\varepsilon} \in B$. Thus, the conditions of Lemma 2.12 are satisfied.

Write $B = \{\alpha^\beta \cdot \alpha^\varepsilon \mid \alpha^\varepsilon < \alpha^\gamma\}$ and $A = \{\alpha^\beta \cdot \zeta \mid \zeta < \alpha^\gamma\}$. Clearly $B \subseteq A$. Let $\alpha^\beta \cdot \zeta \in A$. Let $\varepsilon = \min\{\eta \mid \alpha^\eta \geq \zeta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\alpha^\eta < \zeta$. Then $\zeta < \alpha^\gamma = \sup_{\eta<\gamma} \alpha^\eta \leq \zeta \lightning$. Hence, $\varepsilon < \gamma$, i.e. $\alpha^\varepsilon < \alpha^\gamma$. By construction, $\zeta \leq \alpha^\varepsilon$. Thus, by Left-Monotonicity, $\alpha^\beta \cdot \zeta \leq \alpha^\beta + \alpha^\varepsilon \in B$. Thus, the conditions of Lemma 2.12 are satisfied. □

5. The Cantor Normal Form

Lemma 5.1. *If $\alpha < \beta$ and $n, m \in \omega \setminus \{0\}$, $\omega^\alpha \cdot n < \omega^\beta \cdot m$.*

Proof. $\alpha + 1 \leq \beta$, so $\omega^{\alpha+1} \leq \omega^\beta$ by Left-Monotonicity (of exponentiation). Hence (by Left-Monotonicity of multiplication), $\omega^\alpha \cdot n < \omega^\alpha \cdot \omega = \omega^{\alpha+1} \leq \omega^\beta \leq \omega^\beta \cdot m$. □

Lemma 5.2. *If $\alpha_0 > \alpha_1 > \ldots > \alpha_n$, and $m_1, \ldots, m_n \in \omega$, then $\omega^{\alpha_0} > \sum_{1\leq i\leq n} \omega^{\alpha_i} \cdot m_i$.*

Proof. If any $m_i = 0$ it may just be omitted from the sum. So suppose all $m_i > 0$. $n = 0$ and $n = 1$ are the trivial cases. Consider $n = 2$:

$\omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_2} \cdot m_2 \leq \omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_1} \cdot m_1$ by the lemma above and Left-Monotonicity of addition. Then again by the previous lemma $\omega^{\alpha_1} \cdot m_1 \cdot 2 < \omega^{\alpha_0}$.

Continue via induction: Suppose the lemma holds for n. Then consider the sequence $\alpha_1, \ldots, \alpha_n$. It follows that $\sum_{2 \leq i \leq n+1} \omega^{\alpha_i} \cdot m_i < \omega^{\alpha_1}$. By the $n = 2$ case, $\omega^{\alpha_0} > \omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_1}$ and by Left-Monotonicity of addition, $\omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_1} > \sum_{1 \leq i \leq n} \omega^{\alpha_i} \cdot m_i$. □

Theorem 5.3 (Cantor Normal Form (CNF))**.** *For every ordinal α, there is a unique $k \in \omega$ and unique tuples $(m_0, \ldots, m_k) \in (\omega \setminus \{0\})^k$, $(\alpha_0, \ldots, \alpha_k)$ of ordinals with $\alpha_0 > \ldots > \alpha_k$ such that:*

$$\alpha = \omega^{\alpha_0} \cdot m_0 + \ldots + \omega^{\alpha_k} \cdot m_k$$

Proof. Existence by induction on α: If $\alpha = 0$, then $k = 0$. Suppose that every $\beta < \alpha$ has a CNF. Let $\hat{\alpha} = \sup\{\gamma \mid \omega^\gamma \leq \alpha\}$ and let $\hat{m} = \sup\{m \in \omega \mid \omega^{\hat{\alpha}} \cdot m \leq \alpha\}$. Note that $\omega^{\hat{\alpha}} \leq \alpha$: If not, then $\alpha \in \omega^{\hat{\alpha}}$. Then there is γ, $\omega^\gamma \leq \alpha$ with $\alpha \in \gamma$. But since $\omega^{\alpha+1} > \omega^\alpha \geq \alpha$, $\gamma < \alpha + 1$, i.e. $\gamma \leq \alpha$↯.

Also note that $\hat{m} \in \omega$: If not, then $\hat{m} = \omega$, hence: $\alpha < \omega^{\hat{\alpha}+1} = \omega^{\hat{\alpha}} \cdot \omega = \sup_{n \in \omega} \omega^{\hat{\alpha}} \cdot n \leq \alpha$↯.

By construction, $\omega^{\hat{\alpha}} \cdot \hat{m} \leq \alpha$, so there is $\varepsilon \leq \alpha$ with $\alpha = \omega^{\hat{\alpha}} \cdot \hat{m} + \varepsilon$. Show that $\varepsilon < \alpha$: Suppose not, then $\varepsilon \geq \alpha$, hence $\varepsilon \geq \omega^{\hat{\alpha}}$, so there is $\zeta \leq \varepsilon$ with $\varepsilon = \omega^{\hat{\alpha}} + \zeta$, i.e. $\alpha = \omega^{\hat{\alpha}} \cdot \hat{m} + \omega^{\hat{\alpha}} + \zeta$. By left-distributivity, $\alpha = \omega^{\hat{\alpha}} \cdot (\hat{m} + 1) + \zeta \geq \omega^{\hat{\alpha}} \cdot (\hat{m} + 1)$, contradicting the choice of $\hat{m}$.

Thus, by induction, ε has a CNF $\sum_{i \leq l} \omega^{\beta_i} \cdot n_i$. Note that $\beta_0 \leq \hat{\alpha}$: If not, $\beta_0 > \hat{\alpha}$, i.e. by the choice of $\hat{\alpha}$, $\omega^{\beta_0} > \alpha$, so $\varepsilon \geq \omega^{\beta_0} > \alpha$↯.

Now state the CNF of α: If $\beta_0 < \hat{\alpha}$ set $k = l + 1$, $\alpha_0 = \hat{\alpha}$, $m_0 = \hat{m}$ and $\alpha_i = \beta_{i-1}$, $m_i = n_{i-1}$ for $1 \leq i \leq k$. And if $\beta_0 = \hat{\alpha}$ set $k = l$, $m_0 = n_0 + \hat{m}$, $\alpha_0 = \hat{\alpha}$ and $\alpha_i = \beta_i$, $m_i = n_i$ for $1 \leq i \leq k$.

Uniqueness: Suppose not and let α be the minimal counterexample. Let $\alpha = \omega^{\alpha_0} \cdot m_0 + \ldots + \omega^{\alpha_m} \cdot m_m = \omega^{\beta_0} \cdot n_0 + \ldots + \omega^{\beta_n} \cdot n_n$. Obviously $\alpha > 0$, i.e. the sums are not empty.

Show $\alpha_0 = \beta_0$: Suppose not, wlog assume $\alpha_0 > \beta_0$. Consider the previous lemma. Then $\alpha \geq \omega^{\alpha_0} \cdot m_0 > \omega^{\beta_0} \cdot n_0 + \ldots + \omega^{\beta_n} \cdot n_n = \alpha$↯.

Then show $m_0 = n_0$: Suppose not, wlog assume $m_0 < n_0$. Then, again by the previous lemma, $\omega^{\alpha_0} > \sum_{1 \leq i \leq m} \omega^{\alpha_i} \cdot m_i$. So, by Left-Monotonicity of addition, $\alpha < \omega^{\alpha_0} \cdot m_0 + \omega^{\alpha_0}$, i.e. $\alpha < \omega^{\alpha_0} \cdot (m_0 + 1) \leq \omega^{\alpha_0} \cdot n_0 \leq \alpha$↯.

So $\omega^{\alpha_0} \cdot m_0 = \omega^{\beta_0} \cdot n_0$, so by Left-Monotonicity, $\omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_m} \cdot m_m = \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_n} \cdot n_n$. These terms are strictly smaller than α by the previous lemma. By minimality of α, $m = n$, and the α's, β's, m's and n's are equal. Thus α has a unique CNF↯. □