Supplement to “Weak Rejection”

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Proofs of the rules for rejected $\wedge$ and $\vee$

($-\vee$I.) If $\vdash^+ -A$ and $\vdash^+ -B$, then $+A \vdash^+ \bot$ and $+B \vdash^+ \bot$. Thus $+(A \vee B) \vdash \bot$ by ($+\vee$E.). Hence $-(A \vee B)$ by Smilean reductio. $\square$

($-\vee$E.) $-(A \vee B), +A \vdash +(A \vee B)$ by ($+\vee$I.). Hence $-(A \vee B), +A \vdash \bot$ by (Rejection). And thus $-(A \vee B) \vdash -A$ by Smilean reductio. $\square$

($-\vee$E.$_2$) is analogous. $\square$

($-\wedge$I.$_1$) $-A, +(A \wedge B) \vdash +A$ by ($+\wedge$E.), so $-A, +(A \wedge B) \vdash \bot$ by (Rejection). Thus $-A \vdash -(A \wedge B)$ by (SR$_1$). $\square$

($-\wedge$I.$_2$) is analogous. $\square$

($-\wedge$E.) Suppose $\varphi = +C$. Assume $-A \vdash +C$ and $-B \vdash +C$.
Then $-(A \wedge B), -A, -C \vdash \bot$ and $-(A \wedge B), -B, -C \vdash \bot$ by (Rejection).
Hence $-(A \wedge B), -C \vdash +A$ and $-(A \wedge B), -C \vdash +B$ by (SR$_2$).
So, $-(A \wedge B), -C \vdash +(A \wedge B)$ by ($+\wedge$I.), *i.e.*, $-(A \wedge B) \vdash +C$ by (SR$_2$).
The argument for $\varphi = -C$ is analogous by using (SR$_1$) instead of (SR$_2$) in the proof above. $\square$

Proof of (6)

To show: $+(A \to B), -B \vdash -A$.

By Smilean reductio, it suffices to show that $+(A \to B), +A, -B \vdash \perp$.

By $(+\to\text{E.})$, $+(A \to B), +A \vdash +B$. And $+B, -B \vdash \perp$ by (Rejection).

Thus $+(A \to B), +A, -B \vdash \perp$. □

Proof of Lemma 3.3 (Contraposition)

- To show: $+(\neg A \to \neg B), +B \vdash +A$.

  $+(\neg A \to \neg B), -A \vdash -B$ by (SI).

  $+(\neg A \to \neg B), +B, -A \vdash \perp$ by (Rejection).

  $+(\neg A \to \neg B), +B \vdash +A$ by (SR_2). □

Proof of Proposition 3.4 (Classical Negation)

- Double Negation Introduction (DNI): $\vdash +(A \to \neg\neg A)$.

  $+A, \vdash -\neg A$ by $(-\neg\text{I.})$.

  $+A, \vdash +\neg\neg A$ by $(+\neg\text{I.})^*$.

  $\vdash +(A \to \neg\neg A)$ by $(+\to\text{I.})^*$. □

- Double Negation Elimination (DNE): $\vdash +(\neg\neg A \to A)$.

  $\vdash +(\neg A \to \neg\neg\neg A)$ (DNI).

  $\vdash +\neg\neg A \vdash +A$ by Contraposition.

  $\vdash +(\neg\neg A \to A)$ by $(+\to\text{I.})^*$. □

- Law of Non-Contradiction (LNC): $\vdash +\neg(A \wedge \neg A)$.

  We show that $(A \wedge \neg A)$ is absurd to assert, and since this inference requires no further premises, $(+\neg\text{I.})^*$ allows us to infer a negation.

  $+(A \wedge \neg A) \vdash +A$ by $(+\wedge\text{E.})$.

  $+(A \wedge \neg A) \vdash +\neg A$ by $(+\wedge\text{E.})$.

  $+(A \wedge \neg A) \vdash -A$ by $(+\neg\text{E.})$.

$+(A \wedge \neg A) \vdash \bot$ by (Rejection).

$\vdash -(A \wedge \neg A)$ by (SR_1).

$\vdash +\neg(A \wedge \neg A)$ by $(+\neg\mathrm{I.})^*$. $\square$

- Classical Reductio: $\vdash +((A \to (B \wedge \neg B)) \to \neg A)$.
  - By LNC and $(+\to\mathrm{E.})$, $+A, +(A \to (B \wedge \neg B)) \vdash \bot$.
  - By Smilean Reductio, $+(A \to (B \wedge \neg B)) \vdash -A$.
  - By $(+\neg\mathrm{I.})^*$, $+(A \to (B \wedge \neg B)) \vdash +A$.
  - By $(+\to\mathrm{I.})^*$, $\vdash +((A \to (B \wedge \neg B)) \to A)$. $\square$

Proof of Theorem 3.5 (Classicality)

$\Leftarrow$ Suppose $+A \vdash^{D} +B$. We want to transform the proof D into a proof in classical propositional logic. First note that by $(+\neg\mathrm{I.})^*$ any step in the proof with a rejected premise $-C$ can be replaced by $+\neg C$. Further note all rules of WRL express classical valid forms of reasoning when mapping $+C \mapsto C$ and $-C \mapsto \neg C$. Thus, applying this mapping to D yields a classically valid proof.

$\Rightarrow$ Since the rules for asserted (plussed) $\vee$, $\wedge$ and $\to$ are classical and Proposition 3.4 shows that negation is classical, any proof in classical logic can be reproduced on the plussed fragment of weak rejectivist logic. $\square$

Proofs of the rules for rejected $\to$

$(-\to\mathrm{I.})$ $+A, -B, +(A \to B) \vdash +B$ by $(+\to\mathrm{E.})$, so $+A, -B, +(A \to B) \vdash \bot$ by (Rejection). Thus $+A, -B \vdash -(A \to B)$ by (SR_1). $\square$

$(-\to\mathrm{E.}_1)$ $-(A \to B), +B \vdash +(A \to B)$ by $(+\to\mathrm{I.})^*$, so $-(A \to B), +B \vdash \bot$ by (Rejection). Thus $-(A \to B) \vdash -B$ by (SR_1). $\square$

$(-\to\mathrm{E.}_2)$ This follows non-elementarily from the Clasicality theorem:

If $\vdash^{+} -(A \to B)$, then $+\neg(A \to B)$ by $(+\neg\mathrm{I.})^*$. Then $+A$ because $\to$ is a material conditional. $\square$

Proof that Smilean inference is equivalent to $(-\neg\mathrm{E.})^*$

$\Leftarrow$ By $(+\to\mathrm{I.})^*$, the antecedent of (SI) can be written as $+(\neg A \to \neg B), -A$. Then:

$+(\neg A \to \neg B), +\neg A \vdash +\neg B$ by $(+\to\mathrm{E.})$.

$+(\neg A \to \neg B), +\neg A \vdash -B$ by $(+\neg\mathrm{E.})$.

$+(\neg A \to \neg B), +B, +\neg A \vdash \bot$ by (Rejection).

$+(\neg A \to \neg B), +B \vdash -\neg A$ by (SR_1).

$+(\neg A \to \neg B), +B \vdash +A$ by $(-\neg\mathrm{E.})^*$.

$+(\neg A \to \neg B), -A, +B \vdash \bot$ by (Rejection).

$+(\neg A \to \neg B), -A \vdash -B$ by (SR_1).

$\Rightarrow$ Smilean inference entails the classical negation theorem. Thus:

If $\vdash^{+} -\neg A$ then $\vdash +\neg\neg A$ by $(+\neg\mathrm{I.})^*$.

Thus $\vdash^{+} +A$ by double negation elimination. $\square$

Proof of Theorem 3.7 (ω-pointed Soundness)

The proof proceeds by induction on the length n of derivations. We write $\Gamma \vdash^{D} \varphi$ if D is a derivation of φ from the premises in Γ. The base clause, $n = 1$ is trivial as $\Gamma \vdash^{D} \varphi$ with $|D| = 1$ only if $\varphi \in \Gamma$.

Now, for each $k \leq n$ and any set of premises Γ, assume that if $\Gamma \vdash^{D} \psi$, $|D| = k$, then $\Gamma \models \psi$. Then assume that $\Gamma \vdash^{D} \varphi$ for some D with $|D| = n + 1$ and show that $\Gamma \models \varphi$.

- $(+ \to \mathrm{I.})^*$. Assume $\Gamma \vdash^{D} \varphi$ by an application of $(+ \to \mathrm{I.})^*$, *i.e.*, $\varphi = +(A \to B)$ for some A, B and $\Gamma \cup \{+A\} \vdash^{D'} +B$ where D' uses only asserted premises from Γ. Let Γ' be the asserted formulae in Γ. Then $\Gamma' \cup \{+A\} \vdash^{D'} +B$. Assume that $\Gamma' \not\models +(A \to B)$. Then there is a model V of Γ' and a point $y \in \omega$ such that $V \models_y A \wedge \neg B$. Construct an ω-pointed model V' where *every* point is y, *i.e.*, for any x and atom p, $V' \models_x p$ iff $V \models_y p$. Because Γ' contains only asserted formulae, $V' \models \Gamma'$. Also, because $V \models_y A$, $V' \models +A$. Hence $V' \models \Gamma' \cup \{+A\}$. By assumption, $V' \models +B$, but by construction $V' \models +\neg B$. Contradiction.

- $(+\neg\text{I.})^*$. Assume $\Gamma \vdash^D +\neg A$ by an application of $(+\neg\text{I.})^*$, *i.e.*, $\Gamma \vdash^{D'} -A$ where D' uses only asserted premises from Γ. Let Γ' be the asserted formulae in Γ. Then $\Gamma' \vdash^{D'} -A$. Assume that $\Gamma' \not\models +\neg A$. Then there is a model V of Γ' and a point $y \in \omega$ such that $V \models_y A$. Construct an ω-pointed model V' where *every* point is y, *i.e.*, for any x and atom p, $V' \models_x p$ iff $V \models_y p$. Because Γ' contains only asserted formulae, $V' \models \Gamma'$. Also, because $V \models_y A$, $V' \models +A$. But by induction, $V' \models -A$. Contradiction.

- Every point in an ω-pointed model is a model of classical propositional logic. Hence the rules $(+\wedge\text{I.})$, $(+\wedge\text{E.}_1)$, $(+\wedge\text{E.}_2)$, $(+\vee\text{I.}_1)$, $(+\vee\text{I.}_2)$, $(+\vee\text{E.})^*$, $(+\to\text{E.})$ are trivially sound. For instance, if $(+\wedge\text{I.})$ is the last rule applied in D, then $\varphi = +(A \wedge B)$ and there are derivations D' and D'' of $+A$ and $+B$ respectively from Γ. By induction hypothesis, $\Gamma \models +A$ and $\Gamma \models +B$. That is, every point in every model of Γ satisfies A and B, hence every point in every model of Γ satisfies $A \wedge B$. Thus $\Gamma \models +(A \wedge B)$.

- $(-\neg\text{I.})$. If there is a derivation of $+A$ from Γ then by induction, all points in all models of Γ satisfy A. Since there is at least one point in every model, $\Gamma \models -\neg A$.

- $(+\neg\text{E.})$. If there is a derivation of $+\neg A$ from Γ then by induction, all points in all models of Γ satisfy $\neg A$. Since there is at least one point in every model, $\Gamma \models -A$.

- (Rejection). By definition, no ω-pointed model can satisfy $+A$ and $-A$. Hence if there are subderivations of $+A$ and $-A$ from Γ, then by induction, Γ has no model, *i.e.*, $\Gamma \models \perp$.

- (SR_1). Assume that the last step in the derivation D is an application of (SR_1) to show $-A$. Then there is a derivation D' of shorter length with $\Gamma, +A \vdash^{D'} \perp$. By induction, $\Gamma \cup \{+A\} \models \perp$, *i.e.*, $\Gamma \cup \{+A\}$ is unsatisfiable. If already Γ is unsatisfiable, then trivially $\Gamma \models -A$. If Γ is satisfiable then let V be any model of Γ. By assumption, $V \not\models +A$, *i.e.*, there is a $x \in \omega$ such that $V \models_x \neg A$, hence $V \models -A$.

- (SR_2). Assume that the last step in the derivation D is an application of (SR_1) to show $+A$. Then there is a derivation D' of shorter length with $\Gamma, -A \vdash^{D'} \perp$. By induction, $\Gamma \cup \{-A\} \models \perp$, *i.e.*, $\Gamma \cup \{-A\}$ is unsatisfiable. If already Γ is unsatisfiable, then trivially $\Gamma \models +A$. If Γ is satisfiable then let V be any model of Γ. By assumption, $V \not\models -A$, *i.e.*, for all $x \in \omega$, $V \models_x A$, hence $V \models +A$.

At this point, we note that since $(+ \rightarrow \mathrm{I.})^*$ is sound, if $+A \overset{+}{\cdots} +B$ appears in a derivation from Γ, then $\Gamma \models +(A \rightarrow B)$.

- (SI). By induction, we know that $\Gamma \models +(\neg A \rightarrow \neg B)$ and $\Gamma \models -A$. Let $x \in V$ such that $V \not\models_x A$. Then, $V \models_x \neg A$. Since $V \models_x (\neg A \rightarrow \neg B)$ then also $V \models_x \neg B$. Thus $V \models -B$.

- The rules on rejected premises are derivative, so we need not show their soundness.

This concludes the induction. $\square$

Proof of Lemma 3.8

Auxiliary lemma. If Γ is a WRL-consistent set of only asserted formulae, then $\Gamma' = \{A \mid +A \in \Gamma\}$ is classically satisfiable.

Proof. Assume Γ' is not classically consistent. Then there is a proof of $\perp$ from the premises in Γ'. By Classicality this proof can be carried out in weak rejectivist logic with the premises from Γ. Therefore, by contraposition, if Γ is WRL-consistent, Γ' is classically consistent. By the satisfiability theorem in classical logic, if Γ is WRL-consistent, Γ' is classically satisfiable.

Proof of 3.8. By the auxiliary, it suffices to show that $\Gamma \cup \{+\neg A\}$ is consistent. Assume it is not. Then $\Gamma, +\neg A \vdash \perp$, *i.e.*, $\Gamma \vdash -\neg A$ by (SR_1). Since Γ only contains asserted formulae, $\Gamma \vdash +\neg\neg A$ by ($+\neg$I.). Hence $\Gamma \vdash +A$ by (DNE), which contradicts the assumption that $\Gamma \cup \{-A\}$ is consistent. $\Box$

Proof of Theorem 3.9 (ω-pointed Completeness)

Show a Model Existence result first. That is, for every consistent Γ there is an ω-pointed model of Γ.

Proof of Model Existence.

Use Lemma 3.8 to construct an ω-pointed model by separating the asserted from the rejected formulae in Γ. Then we let all the asserted formulae hold at every point and let every rejected formula fail at a single point.

So let Γ^+ be the set of asserted formulae in Γ and let $\{-A_i \mid i \in \omega\}$ be an enumeration of the rejected formulae in Γ. For all $i \in \omega$, define $\Gamma^i = \Gamma^+ \cup \{-A_i\}$. As $\Gamma^i \subseteq \Gamma$ for all i, the Γ^i are consistent. By Lemma 3.8 there are interpretations I_i of classical propositional logic that satisfy $\{\neg A_i\} \cup \{A \mid +A \in \Gamma^+\}$. Then we can define an ω-pointed model V by setting $V(x) = \{p \mid I_x(p) = \text{True}\}$. With this construction, clearly, $V \models \Gamma$.

Proof of 3.9. By case distinction on the force of φ:

- Assume $\Gamma \models +A$ and $\Gamma \not\vdash +A$. Then $\Gamma \cup \{-A\}$ is consistent and hence there is a V with $V \models \Gamma \cup \{-A\}$. Then $\Gamma \not\models +A$.

- Assume $\Gamma \models -A$ and $\Gamma \not\vdash -A$. Then $\Gamma \cup \{+A\}$ is consistent and hence there is a V with $V \models \Gamma \cup \{+A\}$. Then $\Gamma \not\models -A$.

Proof that modalised WRL axiomatises the class of KD45 frames. The Classicality result tells us that the propositional logic underlying the modal formulation of weak rejectivist logic is classical (in the same way that the points in an ω-pointed model are classical propositional interpretations). Hence, we only need to check the frame conditions. Letting the modal operators embed allows us to write the Smilean reductios as axiom schemes on modal frames as follows:

$$(\mathrm{SR}_1)\ (\neg\Box A) \to (\Box\neg\Box A) \quad (\mathrm{SR}_2)\ (\neg\Box\neg\Box A) \to (\Box A).$$

So let $M = (W, V, R)$ be a model of the two schemes above. We show that M satisfies the axioms (D), (4) and (5).

- (D). Let $W \in W$ with $M, w \models \Box A$. We need to show that $M, w \models \Diamond A$. Assume this is false. Then $M, w \models \neg\Diamond A$, *i.e.*, $M, w \models \Box\neg A$. By (+¬E.) then $M, w \models \Box\neg\Box A$. Then by (Rejection), $M, w \models \bot$. Contradiction.

- (4). Let $w \in W$ with $M, w \models \Box A$. We need to show that $M, w \models \Box\Box A$. By (Rejection), $\Box A, \Box\neg\Box A \vdash \bot$, hence $M, w \models \neg\Box\neg\Box A$. We can bracket this to read as $M, w \models \neg\Box(\neg\Box A)$ and apply (SR_1) outside the brackets. That is, $M, w \models \Box\neg\Box(\neg\Box A)$. Bracket this as $M, w \models \Box(\neg\Box\neg\Box A)$. Then apply the axiom scheme (SR_2) in the brackets to arrive at $M, w \models \Box\Box A$.

- (5). We need to show $M \models \neg\Box A \to \Box\neg\Box A$. This is exactly (SR_1). $\Box$

Proof of Lemma 3.12.

KD45 modal logic is sound and complete for Kripke frames that are serial, transitive and Euclidean; these properties correspond to axioms (D), (4) and (5), respectively. That is, a model is a triple (W, R, V) where W is a set of worlds, V is a mapping from worlds to sets of atoms and $R \subseteq W^2$ is a relation satisfying:

(S) $\forall x \exists y(xRy)$ (T) $\forall x, y, z(xRy \wedge yRz \rightarrow xRz)$ (E) $\forall x, y, z(xRy \wedge xRz \rightarrow yRz)$.

We now show the lemma by contraposition. Suppose $\Gamma \not\models^{\mathrm{KD45}} \Box(A \rightarrow B)$. Then there is a model $M = (W, R, V)$ and a world $w \in M$ with $M \models \Gamma$ and $M, w \not\models \Box(A \rightarrow B)$, *i.e.*, $M, w \models \Diamond(A \wedge \neg B)$. Let $v \in W$ be a witness for the latter proposition, *i.e.*, wRv and $M, v \models A \wedge \neg B$.

Consider the following model $M' = (\{w, v\}, \{(v, v), (w, v)\}, V \restriction \{w, v\})$. Trivially, M' is serial and satisfies (T) and (E). We now verify that it is also a model of Γ. Let $\varphi \in \Gamma$. Then there is a propositional C such that $\varphi = \Box C$. Since $M \models \Gamma$, $M, w \models \Box C$, hence $M, v \models C$. Thus $M', v \models C$ and therefore $M', w \models \Box C$ and $M', v \models \Box C$. Hence M' is a KD45-model of Γ. Now observe that $M', w \models \Box A$ but $M', w \not\models \Box B$. Hence $\Gamma, \Box A \not\models^{\mathrm{KD45}} \Box B$. □

Proof of Theorem 3.13 (KD45 Soundness).

The previous lemma establishes the soundness of modalised $(+\rightarrow \mathrm{I.})^*$. The only interesting cases here are (Rejection), $\neg$-elimination and -introduction, the Smilean reductios and the Smilean inference rules. The soundness of the other rules can be established as in the proof of soundness on ω-pointed models (Theorem 3.7). We show soundness by assuming that $M = (W, R, V)$ is an arbitrary model of KD45 and show that it satisfies the rules of weak rejectivist logic.

- (Rejection). If $M, w \models \Box A$ and $M, w \models \Box\neg\Box A$, then, by seriality, there is a $v \in W$, wRv, with $M, v \models \neg\Box A$. Then there is $v' \in W$, vRv', and $M, v' \models \neg A$. By Transitivity, also $M, v' \models A$. Hence there is no such M.

- $(+\neg\text{I.})^*$. Assume $\Box\neg\Box A$ holds in all models where the set of only necessitated formulae Γ is true. Let $M = (W, R, V)$ be such a model and assume that there is some w with $M, w \not\models \Box\neg A$, *i.e.*, $M, w \models \Diamond A$. Let $v \in W$ be such that wRv and $M, v \models A$. Let $M' = (W', R', V')$ be a model with $W' = \{w, v\}$, $R' = \{(w, v), (v, v)\}$ and $V'(v) = V(v)$. R' clearly is serial, transitive and Euclidean. Since all formulae in Γ are of the form $\Box B$ for propositional B and $M, w \models \Gamma$, $M, v \models B$ for all B with $\Box B \in \Gamma$. Hence also $M', w \models \Gamma$, as the valuation on v is the same. But by construction also $M', v \models A$ and vRv, so $M, w \models \Box\Box A$. This contradicts the initial assumption that $\Box\neg\Box A$ is true in all models of Γ.

- $(+\neg\text{E.})$. If $M, w \models \Box\neg A$, then $M, w \models \Diamond\neg A$ by (D). Then, trivially, $M, w \models \Box\neg\Box A$.

- $(-\neg\text{I.})$. Analoguous to $(+\neg\text{E.})$.

- (SI). Assume $M, w \models \Box(\neg A \to \neg B)$ and $M, w \models \Box\neg\Box A$. The latter formula can be put as $M, w \models \Box\Diamond\neg A$. By Transitivity, it also holds that $M, w \models \Box\Box(\neg A \to \neg B)$. Hence $M, w \models \Box\Diamond\neg B$, which is equivalent to $M, w \models \Box\neg\Box B$.

We show the soundness of the Smilean reductios by deriving them from the KD45-axioms. Modal logic satisfies a version of (CNI) not expressible in bilateralist logics, namely $\Gamma, \varphi \vdash \bot \Rightarrow \Gamma \vdash \neg\varphi$. This cannot be put in the bilateralist language, because the force markers cannot embed under a negation sign.

- (SR_1). Suppose $\Gamma, \Box A \vdash \bot$. Then $\Gamma \vdash \neg\Box A$, hence by (5), $\Gamma \vdash \Box\neg\Box A$.

- (SR_2). Suppose $\Gamma, \Box\neg\Box A \vdash \bot$. Then $\Gamma \vdash \neg\Box\neg\Box A$. By the contrapositive of (5), $\Gamma \vdash \neg\neg\Box A$, *i.e.*, $\Gamma \vdash \Box A$. $\Box$

Proof of Theorem 3.15 (KD45 Completeness).

To prove this, we will use the completeness result on ω-pointed models. We show that ω-pointed models (where the force markers $+$ and $-$ are considered non-embeddable) embed into KD45-models. That is, we can transform ω-pointed models into models of KD45 modal logic to show the non-embedded completeness theorem above.

Auxiliary Lemma. Let V be a ω-pointed model. Then there is a KD45-model M^V such that $V \models +A$ iff $M^V \models \Box A$ and $V \models -A$ iff $M^V \models \Box\neg\Box\varphi$.

Proof. Define M^V as follows: $M^V = (\omega, \omega^2, V)$. Trivially, the relation is serial, transitive and Euclidean. Now observe: $V \models +A$ iff $\forall n : V \models_n A$ iff $M^V \models \Box A$. $V \models -A$ iff $\exists n : V \models_n \neg A$ iff $M^V \models \Diamond\neg A$ iff $M^V \models \neg\Box A$ iff $M^V \models \Box\neg\Box A$. $\Box$

Proof of 3.15. Let Γ be a set of rejectivist modal formulae. We show that if $\Gamma \models^{\mathrm{KD45}} \varphi$ then $\Gamma^r \vdash \varphi^r$ in weak rejectivist logic. Assume not, *i.e.*, $\Gamma \models^{\mathrm{KD45}} \varphi$ and $\Gamma^r \nvdash \varphi^r$.

- Case 1: $\varphi = \Box A$. Then $\varphi^r = +A$. By assumption, $\Gamma^r \cup \{-A\}$ is consistent. By ω-pointed Model Existence, there is an ω-pointed model V of the set $\Gamma^r \cup \{-A\}$. By the previous lemma, there is a KD45-model M^V such that $M^V \models \Gamma$ and $M^V \models \Box\neg\Box A$. This contradicts the assumption that $\Gamma \models^{\mathrm{KD45}} \varphi$, because then also $\Gamma \models^{\mathrm{KD45}} \Box\Box A$.
- Case 2: $\varphi = \Box\neg\Box A$. Then $\varphi^r = -A$. By assumption, $\Gamma^r \cup \{+A\}$ is consistent. As above, there is an ω-pointed model V of $\Gamma^r \cup \{+A\}$. By the previous lemma, there is a KD45-model M^V such that $M^V \models \Gamma$ and $M^V \models \Box A$. Then also $M^V \models \Box\Box A$. This contradicts the assumption that $\Gamma \models^{\mathrm{KD45}} \varphi$. $\Box$

Proof of Theorem 4.1.

There are five other rules in bilateralist logic.

- $(-\neg\mathrm{E.})$. If $-\neg A$ then by $(+\neg\mathrm{I.})$ $+\neg\neg A$, hence $+A$ by DNE.
- $(-\to\mathrm{E.}_2)$. From $-(A \to B)$, $(+\neg\mathrm{I.})$ infers $+\neg(A \to B)$. Since the logic on $+$ is classical, this is $+(A \wedge \neg B)$. Hence $+A$ by $(+\wedge\,\mathrm{E.})$.
- $(-\vee\mathrm{I.})$. From $-A$ and $-B$, $(+\neg\mathrm{I.})$ infers $+\neg A$ and $+\neg B$, *i.e.*, $+(\neg A \wedge \neg B)$. Since the logic on $+$ is classical, this is $+\neg(A \vee B)$. Hence by $(-\neg\mathrm{I.})$, $-A \vee B$.

The rules $(+\vee\mathrm{E.})^*$ and $(+\to\mathrm{I.})^*$ differ from the strong bilateralist rules by only allowing asserted premises in their subderivations. In the presence of $(-\neg\mathrm{E.})$ and $(+\neg\mathrm{I.})$ this does not change the proof-theoretic strength of the calculus. $\square$